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STRONG TRACES FOR DEGENERATE PARABOLIC-HYPERBOLIC EQUATIONS AND APPLICATIONS

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Dedicated to my love Heejin.

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STRONG TRACES FOR DEGENERATE PARABOLIC-HYPERBOLIC EQUATIONS AND APPLICATIONS

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We consider bounded weak solutions u of a degenerate parabolic -hyperbolic equation defined in a subset $]0,T[\times\Omega \subset \mathbb{R}^+ \times \mathbb{R}^d$. We define a strong notion of trace at the boundary $]0,T[\times\partial\Omega$ reached by L^1 convergence for a large class of functionals of u. Such functionals depend on the flux function of the degenerate parabolic-hyperbolic equation and on the boundary. We also prove the well-posedness of the entropy solution for scalar conservation laws with a strong boundary condition with the above trace result as applications.

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Chapter 1

Introduction

In this chapter we introduce the meaning of entropy solutions, the basic results of the initial boundary value problems for scalar conservation laws and degenerate parabolic-hyperbolic equations, and finally the set-up of our problem.

1.1 Entropy solutions of the Cauchy problems

In this section we discuss entropy solutions for scalar conservation laws and the *Kružkov*'s result as the first step of degenerate parabolic-hyperbolic equations. We consider the scalar conservation laws with an initial condition u_0:

$$\partial_t u + \mathrm{div}_x A(u) = 0, \quad u(0,x) = u_0(x), \quad x \in \mathbb{R}^d \tag{1.1}$$

where $A \in [C^2(\mathbb{R})]^d$. Let us first examine the case of linear fluxes such that $A(u) = uC$ where C is a constant vector. Then we have the classical unique solution for the Cauchy problem (1.1)

$$u(t,x) = u_0(x - tC).$$

We define the characteristics curves to investigate the case of non-near fluxes

$$X'(t) = A'(u(t,X)).$$

Let us next calculate u with respect to t along the given characteristics curves. Then we obtain

$$\frac{d}{dt}u(t, X(t)) = \partial_t u + X'(t)\partial_x u(t, X) = (\partial_t u + \mathrm{div}_x A(u))(t, X) = 0.$$

Thus, by the implicit function theorem, the Cauchy problem (1.1) has the unique solution near 0, but it can globally have several solutions. The precise proposition follows.

Proposition 1.1.1. *Assume that u_0, defined in $\mathbb{R}^d$, is bounded and Lipschitz continuous. Let*

$$\kappa = essinf\{\mathrm{div}_x A'(u_0(y)) | y \in \mathbb{R}^d\}.$$

Then there exists a classical solution u of (1.1) on $[0, -\frac{1}{\kappa})$. Furthermore, if u_0 is of C^k, so is u.

Notice that there exist several solutions for $t > -\frac{1}{\kappa}$. Let us now construct a Cauchy problem which admits more than one weak solution.

Example 1.1.1. *(Burger's equation)*

$$\partial_t u + \partial_x \frac{u^2}{2} = 0,$$

with the initial condition

$$u(0, x) = \begin{cases} -1, & x < 0 \\ 1, & x > 0 \end{cases}$$

Burger's equation (1.1.1) admits infinitely many weak solutions

$$u_p(t,x) = \begin{cases} -1, & -\infty < x \le -t \\ \frac{x}{t}, & -t < x \le -pt \\ -p, & -pt < x \le 0 \\ p, & 0 < x \le pt \\ \frac{x}{t}, & pt < x \le t \\ 1, & t < x < \infty \end{cases}$$

for any $p \in [0,1]$. To resolve the issue of nonuniqueness, we impose the following weak solution.

Definition 1.1.1. *We say that a weak solution of (1.1) is an entropy solution if it verifies the following inequalities for any convex function η such that $q' = \eta' A'$:*

$$\partial_t \eta(u) + \mathrm{div}_x q(u) \le 0, \quad in \quad \mathcal{D}'. \tag{1.2}$$

We notice that (1.2) means

$$\int_Q (\eta(u)\varphi_t + q(u) \cdot \nabla_x \varphi) dxdt \ge 0 \tag{1.3}$$

for all $\varphi \in C_c^\infty(]0,T[\times \mathbb{R}^d)$ with $\varphi \ge 0$ where $Q =]0,T[\times \mathbb{R}^d$.

In the case of scalar conservation laws, the initial value problem in $\mathbb{R}^+ \times \mathbb{R}^d$ was first studied by *Kružkov* [18] where the existence and uniqueness of solutions whenever the initial value is reached strongly in $L^1_{loc}(\mathbb{R}^d)$ was shown. His work follows.

Theorem 1.1.1. *(Kružkov[18]) Let $u_0 \in L^\infty(\mathbb{R}^d)$. Then there exists one and only one entropy solution of (1.1) and (1.2) in $L^\infty(Q) \bigcap C([0,T); L^1_{loc}(\mathbb{R}^d))$.*

Furthermore, if u_0, v_0 are the initial values for entropy solutions u, v of (1.1) and (1.2) respectively, then for any $t > 0$ and $r > 0$,

$$\int_{B_r} |u(t,x) - v(t,x)|dx \leq \int_{B_{r+m}} |u_0(x) - v_0(x)|dx,$$

where $B_R = \{x \in \mathbb{R}^d ||x| < R\}$ and $m = \sup\{|A'(s)||s \in [\inf(u_0, v_0), \sup(u_0, v_0)]\}$.

1.2 Initial boundary value problems for scalar conservation laws

In this section we consider the following Cauchy problem

$$\partial_t u + \mathrm{div}_x A(u) = 0, \quad u(0,x) = u_0(x), \quad (t,x) \in Q_T =]0,T[\times\Omega \tag{1.4}$$

where $A \in [C^2(\mathbb{R})]^d$ and $\partial\Omega$ is of C^k. Unfortunately, the boundary condition for some examples of (1.4) can not be described on any part of the boundary $]0,T[\times\partial\Omega$. Here an example follows.

Example 1.2.1. *Let us take the following transport equation.*

$$\partial_t u + a\partial_x u = 0, \quad (t,x) \in]0,1[\times]0,1[, \tag{1.5}$$

with the initial condition

$$u(0,x) = \sin\frac{1}{x}$$

Notice that if $a = 0$, $u(t,x) = \sin\frac{1}{x}$ is obviously a solution and it is not defined on $\{0\}\times]0,1[$. To resolve this problem we need a suitable condition on the boundary Γ. Bardos, Le Roux, and Nedelec [2] have first studied this issue for the initial boundary value problem of scalar conservation laws with the assumption $u \in BV$ and they proposed the following appropriate boundary condition on $\Gamma =]0, T[\times \partial\Omega$:

$$sign(u^\tau - u_b)(A(u^\tau) - A(k)) \cdot \hat{n} \geq 0 \tag{1.6}$$

for $k \in [min\{u^\tau, u_b\}, max\{u^\tau, u_b\}]$ where $\hat{n}$ is the outward unit normal vector to $\partial\Omega$ and u_b is a boundary value to (1.4). They have also shown the well-posedness of (1.2) and (1.4) with the above boundary condition (1.6). Their result follows.

Theorem 1.2.1. *(Bardos, Leroux, Nédélec [2]) Let $u_0 \in L^\infty(\Omega) \bigcap BV(\Omega)$ be some initial data and $u_b \in L^\infty(\Gamma) \bigcap BV(\Gamma)$ boundary data. Then then there exists one and only one entropy solution of (1.2) and (1.4) in $L^\infty(Q) \bigcap C([0,T); L^1_{loc}(\Omega))$. Furthermore, if u_0, v_0 are the initial data and u_b, v_b are the boundary data for entropy solutions u, v of (1.2) and (1.4) respectively, then for any $t > 0$,*

$$\|u(t) - v(t)\|_{L^1(\Omega)} \leq \|u_0 - v_0\|_{L^1(\Omega)} + M\|u_b - v_b\|_{L^1(\Gamma)}$$

where $M = \sup\{\|A'(s)\| | s \in \mathbb{R}\}$.

In [25], Otto has extended the result Theorem 1.2.1 without using the bounded variation of solutions, but with the following condition:

$$\operatorname*{esslim}_{s\to 0} \int_\Gamma G(u(r + s(0, \hat{n}(r))), u_b(r)) \cdot \hat{n}(r)\phi(r)dr \geq 0 \tag{1.7}$$

for all nonnegative $\phi \in C_c^\infty(]0, T[\times\partial\Omega)$ and boundary entropy pairs (P, G) verifying: for every $w \in \mathbb{R}$, $(P(\cdot, w), G(\cdot, w))$ is an entropy pair with $P(w, w)$, and $\partial_z P(w,w) = 0$ and $G(w,w) = 0$. Here he gives a very elegant weak formulation of the boundary condition for this problem which provides well-posedness, but leaves open questions on the structure of the solution at the boundary Γ.

1.3 Degenerate parabolic-hyperbolic equations

In this section we discuss the initial boundary value problem of the degenerate parabolic-hyperbolic equation studied by Carrillo, Mascia, Porretta, Terracina. This kind of problem was first studied by Carrillo [5], but in this section we will introduce Mascia, Porretta, and Terracina [23]. Let us denote an open subset $Q =]0,T[\times\Omega \subset \mathbb{R}^{d+1}$ with C^2 boundary $\partial\Omega$. We consider the following Dirichlet problems

$$\partial_t u + \mathrm{div}_x A(u) = \triangle_x b(u), \quad u(0,x) = u_0(x), \tag{1.8}$$

where $(t,x) \in Q =]0,T[\times\Omega$, the flux function A is in $[C^2(\mathbb{R})]^d$, $u_0 \in L^1(\Omega) \bigcap L^\infty(\Omega)$, and we assume that a nondecresing function b satisfies $b \in C^1(\mathbb{R})$. In this paper we assume the entropy condition which states that, for any convex function η,

$$\partial_t \eta(u) + \mathrm{div}_x q(u) - \mathrm{div}_x(\eta'(u)\nabla_x b(u)) \le 0 \ \ in \ \ \mathcal{D}', \tag{1.9}$$

where $q'(u) = \eta'(u)A'(u)$. Note that a scalar conservation law ($b'(\xi) = 0$) is a particular case of (1.8). From now on we assume the regularity of solutions of (1.8) and (1.9) on $\Gamma =]0,T[\times\partial\Omega$:

$$b(u) \in L^2(0,T;H^1(\Omega)). \tag{1.10}$$

Notice that (1.10) is a typical condition(see [5],[23]). Next we set the weak formulation of the boundary condition to introduce Mascia, Porretta, Terracina's work [23]. Let us denote

$$B(u,v) = sign(u-v)(A(u)-A(v)) - \nabla_x |b(u)-b(v)|$$

and set

$$H(u,v,u_b) := B(u,v) + B(u,u_b) - B(u_b,v).$$

Then they proposed the weak formulation of the boundary condition in the spirit of Otto (1.7).

$$\lim_{\delta\to 0} \frac{1}{\delta} \int_0^\delta \int_\Gamma H(u\circ\psi_s(r), v, u_b(r)) \cdot (n_s \circ \psi_s)(r)\zeta(r)d\sigma(r)ds \geq 0 \tag{1.11}$$

where ψ_s is any regular deformation as given in Chen and Frid [7] and $d\sigma$ is the volume element of Γ. They have proved the well-posedness of entropy solutions of (1.8), (1.9), and (1.11), with a very elegant boundary condition (1.11). Thus, we can also address an open question on the structure of the solutions at the boundary Γ. In the following section we discuss this open problem and its historic context.

1.4 Motivations and set up of the problem

Let us next discuss some motivations for the main problem. The question of strong traces also arose initially in the context of the limit of hyperbolic relaxation towards a scalar conservation law in the case $\Omega =]0,+\infty[\times\mathbb{R}^d$, which is the trace at $t=0$ (see for instance Natalini [24], Tzavaras [36]). The question was whether the limit obtained is the one defined by Kruzkov since the strong continuity at $t = 0$ is not naturally preserved at the limit. To avoid any misunderstanding, let us recall that it is well-known that the uniqueness holds if we incorporate the initial value in the inequality in the following way:

$$\int_0^\infty \int_{-\infty}^\infty (\partial_t \phi(t,x)\eta(u(t,x)) + \partial_x \phi(t,x) H(u(t,x)))\, dx\, dt \\ + \int_{-\infty}^\infty \phi(0,x)\eta(u_0(x))\, dx \geq 0,$$

for any non negative test function $\phi \in \mathcal{D}([0,\infty) \times \mathbb{R}^d)$. However this condition gives that $\eta(u(t,\cdot))$ converges to $\eta(u_0)$ when t goes to 0 (at least weakly), which implies the strong convergence of $u(t,\cdot)$ at $t = 0$. Hence, putting the initial value in the entropy inequality is exactly equivalent to assuming the existence of a strong trace at $t = 0$ (i.e. reached by a strong topology).

The first result of this kind has been proven in Vasseur [37] for the system of an isentropic gas with $\gamma = 3$ which has a lot of similarities with the scalar case. It involves the introduction of blow-up techniques and the use of the theory of kinetic formulation of conservation laws introduced by Lions, Perthame, and Tadmor in [21, 22] which allows the use of so-called averaging lemmas ([1, 13, 16, 31]). This blow-up method is inherited from techniques widely used for parabolic equations (see for instance [15]). The method has been generalized in Vasseur [38], in the case of trace for an arbitrary domain Ω of the multidimensional scalar conservation laws involving a "non degenerate" flux function (1.12):

$$\mathcal{L}(\{\xi \mid \tau + \zeta \cdot A'(\xi) = 0\}) = 0, \qquad \text{for every} \quad (\tau,\zeta) \neq (0,0), \tag{1.12}$$

where $\mathcal{L}$ is the Lebesgue measure. We now introduce Vasseur's work, which is one of motivations for our work. To define traces on the boundary, we use the framework of the "regular deformable Lipschitz boundary" (see for instance Chen and Frid in [7], where they consider only Lipschitz boundaries).

Definition 1.4.1. *We say that the set $\partial\Omega$ is a "regular deformable Lipschitz boundary" if it satisfies the following:*

(i): For each $\hat{z} = (\hat{t}, \hat{x}) \in \partial\Omega$, there exists $r_{\hat{z}} > 0$, a Lipschitz mapping $\gamma_{\hat{z}} : \mathbb{R}^d \to \mathbb{R}$ and an isometry for the Euclidean norm $\mathcal{R}_{\hat{z}} : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ such that: upon rotating, relabeling and translating the coordinate axes $(y_0, \hat{y}) = \mathcal{R}_{\hat{z}}(t, x)$ if necessary,

$$\mathcal{R}_{\hat{z}}(\hat{z}) = 0,$$

$$\mathcal{R}_{\hat{z}}(\Omega)\cap]-r_{\hat{z}}, r_{\hat{z}}[^{d+1} = \{y \in]-r_{\hat{z}}, r_{\hat{z}}[^{d+1} \mid y_0 > \gamma_{\hat{z}}(\hat{y})\}.$$

where $Q_{\hat{z}} = \{y \in \mathbb{R}^{d+1} \mid |y_i| \leq r_{\hat{z}}\}$.

(ii): There exists at least one $\partial\Omega$ regular deformation, where, for K an open subset of $\partial\Omega$, we call K a regular deformation if every function $\psi : [0,1] \times K \to \bar{\Omega}$ is bi-Lipschitz homeomorphic over its image and verifies:

(iii): $\psi(0, \cdot) \equiv I_K$, where I_K is the identity map over K.

(iv):

$$\lim_{s \to 0+} \nabla\psi(s, \cdot) \circ \tilde{\gamma}_{\hat{z}} = \nabla\tilde{\gamma}_{\hat{z}} \text{ in } L^1(]-r_{\hat{z}}, r_{\hat{z}}[^{d+1} \cap \mathcal{R}_{\hat{z}}(K)),$$

where for every $\hat{z} \in \partial\Omega$ $\tilde{\gamma}_{\hat{z}}$ is the restriction to $]-r_{\hat{z}}, r_{\hat{z}}[^{d+1} \cap \mathcal{R}_{\hat{z}}(K)$ of the map $\hat{y} \to \mathcal{R}_{\hat{z}}^{-1}(\gamma_{\hat{z}}(\hat{y}), \hat{y})$.

Notice that a smooth boundary $\partial\Omega$ of an open set Ω is also a regular deformable Lipschitz boundary. We call this a regular deformable boundary. Vasseur's result follows:

Theorem 1.4.1. *Let $\Omega \subset \mathbb{R}^{d+1}$ be an open set with a regular deformable Lipschitz boundary and u is a solution of (1.2) and (1.4). Assume that the flux A satisfies (1.12). Then there exists $u^\tau \in L^\infty(\partial\Omega)$ such that for every $\partial\Omega$ regular Lipschitz deformation ψ and every compact set $K \subset\subset \partial\Omega$:*

$$\operatorname*{esslim}_{s\to 0} \int_K |u(\psi(s,\hat{z})) - u^\tau(\hat{z})| d\mathcal{H}^d(\hat{z}) = 0 \tag{1.13}$$

where $d\mathcal{H}^d$ is the d-dimensional Hausdorff measure, In particular, for any continuous function G, $[G(u)]^\tau = G(u^\tau)$.

For this work, he used the Blow-up technique and the averaging Lemma in Souganidis and Perthame [31] with the condition (1.12) on the flux A. This condition (1.12) can be seen as a non-degeneracy property since it avoids flux functions whose restriction to an open subset is linear. This assumption permits the use of the averaging lemmas in this case. The case of a trace at $t = 0$ has been solved recently by Panov [29] for general flux. He shows that any solution verifying (1.1) and (1.2) in $]0,\infty[\times\mathbb{R}^d$ has a trace reached by the strong topology at $t = 0$. The approach involves the blow-up method with refined techniques of H-measures. Let us also mention that those blow-up methods in the framework of a kinetic formulation of conservation laws have also been used in context of the geometric measure theory, using different tools (see for instance the result of De Lellis, Otto and Westdickenberg [12]). For a review of these kind of results, we refer to the books of Dafermos, Serre [11, 32, 33] and Chen, Frid, Torres, and Ziemer [6, 8], and also mention Tadmor, Tao [34] for the related regularity problem of degenerate parabolic-hyperbolic type. In [5], Carrillo has recently shown the well-posedness of entropy solutions for boundary valued problems of degenerate parabolic-hyperbolic equations. In [23], Mascia, Porretta, and Terracina have introduced the concept of an entropy solution to degenerate parabolic-hyperbolic equations and they establish the uniqueness of solutions with a boundary condition in the spirit of Otto [25]. We already reviewed this work in section 1.3. You can also see Chen, Karlsen [9] for the related references. We are now interested in studying the strong trace of solutions for a degenerate parabolic-hyperbolic equation on $\Gamma =]0, T[\times \partial\Omega$ as the result on the hyperbolic case in Kwon and Vasseur [19, 38]. In view of these previous results, we can raise the natural questions:

1. Can we show the result of Theorem 1.4.1 without the condition (1.12) for (1.2) and (1.4) on a general domain $\Omega \subset \mathbb{R}^{d+1}$?

2. Can we show the result of Theorem 1.4.1 for the equations (1.8) and (1.9) with the regularity condition (1.10) on a domain $]0, T[\times \partial\Omega \subset \mathbb{R}^{d+1}$?

Notice that the answer is the main theorem 1.4.2 which will follow on polyhedrons. From now on we assume the domain $\partial\Omega$ to be a polyhedron and $\partial\Omega = \bigcup_{k=1}^{M} \overline{H}_k$ where H_k are the open hyperplanes contained in $\partial\Omega$.

We use the framework of the "regular deformable boundary" (see for instance Chen and Frid in [7]) in order to mention the main result. For domain

Ω, there exists at least one $\partial\Omega$ regular deformation, where, for $\hat{K}$ an open subset of $\partial\Omega$, we call a $\hat{K}$ regular deformation $\hat{\psi} : [0,1] \times \hat{K} \to \bar{\Omega}$ to be homeomorphism, where $\hat{\psi}(0, \cdot) \equiv I_{\hat{K}}$ and $I_{\hat{K}}$ is the identity map over $\hat{K}$. Let us define an open set $K =]0, T[\times \hat{K}$ and a function $\psi(t, x) = (t, \hat{\psi}(x))$. Then, obviously, it is also a Γ regular deformation with an open set K. Let n be the unit outward normal field of $\partial\Omega$ and let us denote $\hat{n}_s(\hat{z})$ the unit outward normal field at $\hat{\psi}(s, \hat{z})$. Let us introduce a deformation ψ on Γ by

$$\psi(s, \hat{t}, \hat{z}) = (\hat{t}, \hat{\psi}(s, \hat{z})) \tag{1.14}$$

for $(\hat{t}, \hat{z}) \in \Gamma$. Notice that $\hat{n}$ is constant on H_k for each $k \in \{1, 2, ..., M\}$, and $\hat{n}_s$ converges strongly to $\hat{n}$ when s goes to 0. To state the main result, we need to introduce a new function, χ-function which comes from the theory of kinetic formulation (See Chen, Lions, Perthame, and Tadmor [10, 21]) and will be discussed in detail in the next chapter. Set $L = \|u\|_{L^\infty(\overline{Q})}$, and introduce a new variable $\xi \in]-L, L[$, denoting for every $v \in]-L, L[$:

$$\chi(v, \xi) = \begin{cases} \mathbf{1}_{\{0 \le \xi \le v\}} & \text{if } v \ge 0 \\ -\mathbf{1}_{\{v \le \xi \le 0\}} & \text{if } v < 0 \end{cases}.$$

Then we introduce new functions called microscopic functions:

Definition 1.4.2. *Let N be an integer, $\mathcal{O}$ be an open set of $\mathbb{R}^N$, $I =]a, b[$ for $-L \le a < b \le L$, and the microscopic function $f \in L^\infty(\mathcal{O} \times I)$ be such that $0 \le \operatorname{sgn}(\xi) f(z, \xi) \le 1$ for almost every (z, ξ). We say that f is a χ-function if there exists $u \in L^\infty(\mathcal{O})$ such that for almost every $z \in \mathcal{O}$ and $\xi \in I$:*

$$f(z, \cdot) = \chi(u(z), \cdot).$$

Notice that, if $0 \in I$ *and* f *is a* χ*-function then we can choose* u *by the formula* $u(z) = \int_I f(z,\xi)\, d\xi$.

More precisely, we want to study the behavior of $u(\psi(s,\cdot))$ as s goes to zero for a deformation ψ as given in (1.14), but we have no information for the limit value of $u(\psi(s,\cdot))$ as s goes to zero at all points except the non-characteristic boundary of Γ. So, we need to consider a more general functional of $u(\psi(s,\cdot))$. Let us denote a function $h(\psi(s,\hat{z}),\xi)$ with the following statement: for every Γ regular deformation ψ,

$$h(\psi(s,\hat{z}),\xi) = A'(\xi)\cdot \hat{n}_s(\hat{z})k(\xi)\mathbb{I}_{\{b'(\xi)=0\}} + h_1(\hat{z},\xi)\mathbb{I}_{\{b'(\xi)>0\}} \tag{1.15}$$

where $k \in L^\infty(\mathbb{R})$, $h_1 \in L^\infty(\Gamma\times]-L,L[)$, and

$$\mathbb{I}_D(\xi) = \begin{cases} 1 & \text{if } \xi \in D \\ 0 & \text{if } \xi \in D^c. \end{cases}$$

for $D \subset \mathbb{R}$. Let us set another new functional G_h given as: for every Γ regular deformation ψ,

$$G_h[u(\psi(s,\hat{z}))] = \int_{-L}^{L} f(\psi(s,\hat{z}),\xi)h(\psi(s,\hat{z}),\xi)d\xi \tag{1.16}$$

where f is a χ-function given by a solution u verifying (1.8) and (1.9). Notice that when we take any convex function η instead of k and $b'(\xi)=0$ for all $\xi \in \mathbb{R}$, $G_h[u(\psi(s,\hat{z}))]$ is the same as the entropy boundary, $[q(u)\cdot\hat{n}_s]^\tau$ mentioned in Kwon and Vasseur [19]. Our main theorem follows:

Theorem 1.4.2. *Let* $\partial\Omega$ *be a polyhedron and* u *be a solution of (1.8) and (1.9) and* f *be a* χ*-function given by the solution* u*. Assume that* b *verifies the* *condition (1.10). Consider h verifying (1.15) and G_h satisfying (1.16). Then, there exists $[G_h u]^\tau \in L^\infty(\Gamma)$ such that for every compact set $K \subset\subset \Gamma$:*

$$\operatorname*{esslim}_{s \to 0} \int_K |G_h u(\psi(s, \hat{z})) - [G_h u]^\tau(\hat{z})| d\sigma(\hat{z}) = 0$$

where $d\sigma$ is the volume element of Γ, and ψ is the deformation defined in (1.14).

Notice that when we take $b'(\xi) = 0$ on $]-L, L[$, the result is the same as that of Kwon and Vasseur [19] for the mono-dimensional scalar conservation laws.

In the case of multidimensional scalar conservation laws, the H-measure tool introduced by Panov seems to be necessary. For the case verifying $b'(\xi) \geq c > 0$, the result follows classical trace results for a parabolic type equation, but the difficulty in the general case lies in the mixing of the two types, parabolic and hyperbolic. To resolve this difficulty we can use the kinetic formulation (Chen and Perthame [10]) as the main tool. We will show that it is enough to do the study, at the kinetic level, on an interval where $b'(\xi)$ is either equal to 0 or uniformly strictly positive. It is strking that this can be done with no extra regularity assumption, the function b. For this, we use extra regularity(BV) in ξ on this kinetic function. The reason is that we need informations on the kinetic formulation (see Chen and Perthame [10]) when we are working on one BV in ξ. It is then enough to gather information on a countable dense set of ξ. This key point was previously introduced in Kwon and Vasseur [19].

The next section is devoted to introducing the mathematical tools that we will use for the proof.

Chapter 2

Mathematical background

2.1 Kinetic formulation

The kinetic formulation was first introduced in Lions, Perthame, and Tadmor [21] for scalar conservation laws. In this section we present the kinetic formulation for degenerate quasilinear parabolic-hyperbolic equations and it allows us to use classical tools of linear analysis to handle the non-linear problems. We first consider the degenerate quasilinear parabolic-hyperbolic equations

$$\frac{\partial}{\partial t}u + \sum_{i=1}^{d} \frac{\partial}{\partial x_i} A_i(u) - \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}[a_{ij}(u) \frac{\partial}{\partial x_j} u] = 0, \quad (t,x) \in Q =]0,T[\times \Omega \quad (2.1)$$

with the initial values

$$u(0,x) = u_0(x) \in L^1(\Omega)$$

where $A : \mathbb{R} \longrightarrow \mathbb{R}^d$ verifies

$$a = A' \in L^\infty_{loc}(\mathbb{R}; \mathbb{R}^d),$$

and we assume that $(a_{ij})_{i,j=1}^d$ is symmetric and locally bounded such that

$$(a_{i,j}) \geq \alpha I_{d \times d} \quad (\alpha \geq 0).$$

We can write a_{ij} by

$$a_{ij}(u) = \sum_{k=1}^{K} \sigma_{ik}(u)\sigma_{kj}(u), \quad \sigma_{ik} \in L^{\infty}_{loc}(\mathbb{R}).$$

The equation satisfies the entropy inequality. To motivate it, we replace $a_{ij}(u)$ by $a_{ij}(u) + \epsilon I$. For any smooth function η, we multiply the equation (2.1) by $\eta'(u^{\epsilon})$. Then we have

$$\frac{\partial}{\partial t}\eta(u^{\epsilon}) + \sum_{i=1}^{d} q_i(u^{\epsilon}) - \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}(\eta'(u^{\epsilon})a_{ij}(u^{\epsilon})\frac{\partial}{\partial_{x_j}}) - \epsilon\triangle\eta(u^{\epsilon}) \tag{2.2}$$

$$= -m_{\epsilon}^{\eta''}(t,x) - n_{\epsilon}^{\eta''}(t,x), \tag{2.3}$$

where the entropy fluxes q_i, are defined by

$$q_i' = a_i\eta_i',$$

the entropy dissipation measure $m_{\epsilon}^{\eta''}(t,x)$ is defined by

$$m_{\epsilon}^{\eta''}(t,x) := \epsilon\eta''(u^{\epsilon})|\nabla u^{\epsilon}|^2 \geq 0,$$

and the parabolic dissipation measure $n_{\epsilon}^{\eta''}(t,x)$ is defined by

$$n_{\epsilon}^{\eta''}(t,x) := \eta''(u^{\epsilon})\sum_{i,j=1}^{d} a_{ij}(u^{\epsilon})\frac{\partial}{\partial_{x_i}}u^{\epsilon}\frac{\partial}{\partial_{x_j}}u^{\epsilon} \geq 0.$$

We now introduce the notations $\beta_{ik}(u)$ and $\beta_{ik}^{\phi}(u)$ for $\phi \in C_0(\mathbb{R})$ with $\phi \geq 0$

$$\beta_{ik}'(u) = \sigma_{ik}(u), \quad (\beta_{ik}^{\phi})'(u) = \sqrt{\phi(u)}\sigma_{ik}(u).$$

Then we get

$$n_{\epsilon}^{\phi}(t,x) := \sum_{k=1}^{K}(\sum_{k=1}^{d}\partial_{x_i}\beta_{ik}^{\phi}(u^{\epsilon}))^2 = \sum_{k=1}^{K}\phi(u^{\epsilon})(\beta_{ik}^{\phi}(u^{\epsilon}))^2$$

We next compute some useful properties to derive a priori bound. For any convex η, we obtain

$$\int_\Omega\int_0^T (m_\epsilon^{\eta''}(t,x)+n_\epsilon^{\eta''}(t,x))dtdx = \int_\Omega\int_0^T \eta''(u^\epsilon)(\sum_{k=1}^K(\sum_{k=1}^d \partial_{x_i}\beta_{ik}^\phi(u^\epsilon))^2 \tag{2.4}$$
$$+\epsilon|\nabla u^\epsilon|^2)dtdx \le \|\eta(u_0)\|_{L^1(\Omega)} \le \|\eta'\|_{L^\infty(\mathbb{R})}\|u_0\|_{L^1(\Omega)}. \tag{2.5}$$

Let us define convenient notations deduced by the duality $(C_0(Q);\mathcal{M}(\mathbb{R}))$,

$$m_\epsilon^\phi(t,x) = \int_{\mathbb{R}} \phi(\xi)m_\epsilon(t,x,\xi)d\xi, \quad n_\epsilon^\phi(t,x) = \int_{\mathbb{R}} \phi(\xi)m_\epsilon(t,x,\xi)d\xi$$

with

$$m_\epsilon(t,x,\xi) = \delta(\xi-u^\epsilon)\epsilon|\nabla u^\epsilon|^2, \quad n_\epsilon(t,x,\xi) = \delta(\xi-u^\epsilon)\sum_{k=1}^K(\sum_{i=1}^d \beta_{ik}^\phi(u^\epsilon))^2.$$

Then we choose the function $\eta(u)=(u-\xi)_+$ for $\xi\ge 0$, or $\eta(u)=(u-\xi)_-$ for $\xi\le 0$, and we obtain

$$\int_0^\infty\int_{\mathbb{R}^d}(m_\epsilon+n_\epsilon)(t,x,\xi)dtdx \le \mu(\xi)\in L_0^\infty(\mathbb{R})$$

where

$$\mu(\xi) := \mathbf{1}_{\{\xi\ge 0\}}\|(u_0-\xi)_+\|_{L^1(\Omega)} + \mathbf{1}_{\{\xi<0\}}\|(u_0-\xi)_-\|_{L^1(\Omega)}.$$

Taking $\eta(u)=\frac{u^2}{2}$, from (2.4), we also have

$$\int_{\mathbb{R}}\int_\Omega\int_0^T (m_\epsilon+n_\epsilon)(t,x,\xi)dtdxd\xi = \int_\Omega\int_0^T (\sum_{k=1}^K(\sum_{i=1}^d \beta_{ik}^\phi(u^\epsilon))^2$$
$$+\epsilon|\nabla u^\epsilon|^2)dtdx \le \frac{1}{2}\|u_0\|_{L^2(\Omega)}.$$

As $\epsilon \longrightarrow 0$, we can deduce that the two measures m, n are non-negative bounded measures which satisfies the kinetic formulation.

Theorem 2.1.1. *(Chen, Perthame [10]) A function $u \in L^\infty(Q)$ with $|u| \leq L$ is a solution of (1.8) and (1.9) in Q if and only if there exists nonnegative measure $m \in \mathcal{M}^+(Q\times]-L, L[)$ such that the related χ-function f defined by $f(t, x, \cdot) = \chi(u(t, x), \cdot)$ for almost every $(t, x, \xi) \in (Q\times]-L, L[)$ verifies:*

$$\partial_t f + A'(\xi) \cdot \nabla_x f - \sum_{i,j=1}^{d} a_{ij}(\xi)\partial_{x_i}\partial_{x_j} f = \partial_\xi m(t, x, \xi) \quad \text{in} \quad \mathcal{D}' \tag{2.6}$$

in $Q\times]-L, L[$.

From now on we will take $a_{ij}(u) = b'(u)\delta^i_j$, where the Kroneker Delta,

$$\delta^i_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

2.2 Definition of H-measure

In this section we introduce the notion of the H-measure which plays an important role in the proof of Proposition 3.1.2. The concept of H-measure was first initiated by Gérard in [14] and Tartar in [35], and here we present the notion of measure valued functions introduced in Panov [28, 29] and study the generalized H-measure which was extended by Panov [28]. Let us first mention some basic useful notations and definitions to define the generalized H-measure.

Definition 2.2.1. *Let Σ be a domain in $\mathbb{R}^d$ and $\mathcal{M}(\Sigma)$ be the space of compactly supported Borel probability measures on $\mathbb{R}$.*

(i)Let $\nu_x \in \mathcal{M}$. Then, we say that $x \to \nu_x$ is a weak measurable map if

$$x \to \int f(s) d\nu_x(s)$$

is measurable on Σ for any continuous function f.

(ii)A measure valued function on Σ is a weakly measurable map $x \to \nu_x$.

(iii)A measure valued function ν_x is said to be bounded if there is a constant $L > 0$ such that supp $\nu_x \subset [-L, L]$ for a.e. $x \in \Sigma$,and $\|\nu_x\|_\infty = inf\{L | supp\ \nu_x \subset [-L, L]\}$.

Let $MV(\Sigma)$ be the collection of bounded measure valued functions on Σ. Then it contains $L^\infty(\Sigma)$.

Definition 2.2.2. *Let $\nu_x^n \in MV(\Sigma)$, $\nu_x \in MV(\Sigma)$.*

(i)the sequence ν_x^n is bounded if there exists a positive $L > 0$ such that $\|\nu_x^n\|_\infty \leq L$ for all n.

(ii)$\nu_x^n \rightharpoonup \nu_x$ (weakly) if for any $g(s) \in C(\mathbb{R})$,

$$\int g(s) d\nu_x^n(s) \rightharpoonup \int g(s) d\nu_x(s) \text{ as } n \to \infty \text{ in } L^\infty(\Sigma) weak - *.$$

(iii)$\nu_x^n \longrightarrow \nu_x$ (strongly) if for any $g(s) \in C(\mathbb{R})$,

$$\int g(s) d\nu_x^n(s) \rightharpoonup \int g(s) d\nu_x(s) \text{ as } n \to \infty \text{ in } L^1_{loc}(\Sigma).$$

We introduce some basic notations. Let $\nu_x^n \in MV(\Sigma)$ be such that $\nu_x^n \rightharpoonup \nu_x \in MV(\Sigma)$ (weakly) and let $x \in \Sigma$ and $p \in \mathbb{R}$, setting:

- $V_n(x,p) = \nu_x^n(]p,\infty[),\ V_0(x,p) = \nu_x(]p,\infty[)$

- $P = \{p_0 \in \mathbb{R} | V_0(x,p) \to V_0(x,p_0)\ as\ p \to p_0\ in\ L^1_{loc}(\Sigma)\}$

- $U_n^p(x) = V_n(x,p) - V_0(x,p)$

- $F(u)(\xi), \xi \in \mathbb{R}^d$, is the Fourier transform

- $S = \{\xi \in \mathbb{R}^d | |\xi| = 1\}$

- $u \to \bar{u}$, u is complex conjugate,

- E is an open set in $\mathbb{R}$.

The notion of extended H-measure introduced in [28] follows:

Theorem 2.2.1. *(See [27–29]) (i) There exists a family of complex valued locally finite Borel measures* $\{\mu^{pq}\}_{p,q\in P}$ *on* $\Sigma \times S$ *and a subsequence* $U_r(x) = \{U_r^p(x)\}_{p\in P}, U_r^p(x) = U_n^p(x)$ *for* $n = n_r$ *such that for every function* ϕ_1 *and* ϕ_2 *belonging to* $C_0(\Sigma)$ *and every function* ψ *belonging to* $C_0(S)$*, one has :*

$$< \mu^{pq}, \phi_1\bar{\phi_2}\psi >= \lim_{r\to\infty} \int_{\mathbb{R}^d} F(\phi_1 U_r^p)(\xi)\overline{F(\phi_2 U_r^q)(\xi)}\psi(\frac{\xi}{|\xi|})d\xi. \qquad (2.7)$$

(ii) The map $(p,q) \to \mu^{pq}$ *is continuous as a map from* $P \times P$ *into the space* $M_{loc}(\Omega)$ *of locally finite Borel measures on* $\Sigma \times S$.

Definition 2.2.3. *We say that the family* $\{\mu^{pq}\}_{p,q\in P}$ *is the H-measure with respect to the subsequence* $\nu_x^r = \nu_x^n, n = n_r$.

Chapter 3

Study on hyperbolic range of b

In this chapter we consider an interval in ξ, denoted by I, on which $b'(\xi) = 0$. We will show the strong trace result on the kinetic function on this range in ξ. This includes the scalar conservation case, but this case is also a key step to showing the general case.

3.1 Reformulation of the problem

We reformulate the problem on a flat boundary of Ω. Let us fix $k \in \{1, 2, ..., M\}$. For each $\hat{x} \in H_k$, we consider an isometry map $\mathcal{R}^k_{\hat{x}} : \mathbb{R}^d \to \mathbb{R}^d$ such that upon rotating, relabeling and translating the coordinate axes $(y_0, \hat{y}) = \mathcal{R}^k_{\hat{x}}(x)$,

$$\mathcal{R}^k_{\hat{x}}(\hat{x}) = 0,$$
$$\mathcal{R}^k_{\hat{x}}(\Omega) \cap] - r_{\hat{x}}, r_{\hat{x}}[^d =]0, r_{\hat{x}}[\times] - r_{\hat{x}}, r_{\hat{x}}[^{d-1}$$

for some $r_{\hat{x}} > 0$.

Thus, for each $\hat{z} = (\hat{t}, \hat{x}) \in \Gamma$, we obtain an isometry map $\Lambda^k_{\hat{z}} : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ given by $\Lambda^k_{\hat{z}}(t, x) = (y_0, t - \hat{t}, \hat{y})$ where $\mathcal{R}^k_{\hat{x}}(x) = (y_0, \hat{y})$. Then we have

$$H_k = \bigcup_{\hat{z} \in H_k} (\Lambda^k_{\hat{z}})^{-1}(\{0\} \times] - r_{\hat{z}}, r_{\hat{z}}[^d).$$

We may assume the above collection of open sets to be countable such that

$$\bigcup_{\hat{z}\in H_k} (\Lambda^k_{\hat{z}})^{-1}(\{0\}\times]-r_{\hat{z}}, r_{\hat{z}}[^d) = \bigcup_{\alpha\in K} (\Lambda^k_{\hat{z}_\alpha})^{-1}(\{0\}\times]-r_{\hat{z}_\alpha}, r_{\hat{z}_\alpha}[^d),$$

where K is a countable set. In order to simplify the notation we write α instead of $\hat{z}_\alpha$ in the indices. Let us define Γ_k by $\Gamma_k =]0,T[\times H_k$. We have

$$\Gamma = \bigcup_{k=1}^{M} \Gamma_k.$$

We denote

$$Q^k_\alpha =]0, r_\alpha[\times]-r_\alpha, r_\alpha[^d,$$

and

$$\Sigma^k_\alpha = (\Lambda^k_\alpha)^{-1}(\{0\}\times]-r_\alpha, r_\alpha[^d).$$

We set $u_\alpha(w) = u((\Lambda^k_\alpha)^{-1}(w))$ where $s = y_0$, $\hat{w} = (t-\hat{t}, \hat{y})$, $g(w,\xi) = \chi(u_\alpha(w,\xi))$, and $A_\alpha(\xi) = \Lambda^k_\alpha(\xi, A(\xi))$. For every fixed α, a deformation ψ defined in (1.14), and every $\hat{w}\in]-r_\alpha, r_\alpha[^d$, we set

$$\tilde{\psi}(s,\hat{w}) = (\Lambda^k_\alpha\circ\psi)(s, (\Lambda^k_\alpha)^{-1}(0,\hat{w})),$$

$$g_\psi(s,\hat{w},\xi) = g(\tilde{\psi}(s,\hat{w}),\xi).$$

From now on we reformulate the problem and construct the weak trace for the case of scalar conservation laws, namely, $b'(\xi) = 0$ on an interval I in the given kinetic formulation (2.6). In this section, we will work only on the above interval despite the fact that g is defined $]-L, L[$. Then, from kinetic formulation (2.6), we obtain the reformulated kinetic equation:

$$a(\xi)\cdot\nabla_w g = \partial_\xi \hat{m} \tag{3.1}$$

for all $\xi \in I$ where $\hat{m}(w,\xi) = m((\Lambda_\alpha^k)^{-1}(w),\xi)$ and $a = A'_\alpha$.

To simplify the notations we keep denoting n_s and n the normal vectors. We will first show that $(a \cdot n_s)g_\psi$ has a (at least weak) trace at $s = 0$ which does not depend on the deformation ψ, namely:

Proposition 3.1.1. *Let I be an interval verifying $b'(\xi) = 0$ for all $\xi \in I$ and g be a solution of (3.1) in $Q_\alpha^k \times I$. Then there exists $a \cdot n g^\tau \in L^\infty(]-r_\alpha, r_\alpha[^d \times I)$ such that for all regular deformation's ψ,*

$$\operatorname*{esslim}_{s\to 0}(a \cdot n_s)g_\psi(s,\cdot,\cdot) = (a \cdot n)g^\tau \text{ in } H^{-1} \cap L^\infty w* .$$

Moreover $(a \cdot n)g^\tau$ is uniquely defined.

This shows the existence of a weak trace on $]-r_\alpha, r_\alpha[^d \times I$ of $(a \cdot n_s)g$ which does not depend on the way chosen to reach the boundary. This result is an extension to Proposition 1 in Vasseur [38]. We give the proof for the sake of completeness.

Proof of Proposition 3.1.1: Since $\|g_\psi(s,\cdot,\cdot)\|_{L^\infty} \leq 1$, by weak compactness and Sobolev imbedding, for every regular deformation ψ and every sequence s^j which tends to 0 there exists a subsequence j_p and a function $h_\psi^\tau \in L^\infty(]-r_\alpha, r_\alpha[^d \times I)$ such that

$$g_\psi(s^{j_p},\cdot,\cdot) \overset{H^{-1}\cap L^\infty W*}{\longrightarrow} h_\psi^\tau \qquad \text{when } j_p \to +\infty. \tag{3.2}$$

Let us now show that $(a \cdot n)h_\psi^\tau$ does not depend on ψ, on the sequence s^j and s^{j_p}. In order to do so, let us first consider $\eta \in \mathcal{D}(\mathbb{R})$:

$$\overline{H}_\eta(w) = \int_I a(\xi)\eta(\xi)g(w,\xi)\, d\xi. \tag{3.3}$$

Multiplying (3.1) by $\eta(\xi)$ and integrating it with respect to ξ we find:

$$\operatorname{div}_w \overline{H}_\eta = -\int_I \eta'(\xi)\hat{m}(w, d\xi) \in \mathcal{M}(]-r_\alpha, r_\alpha[^{d+1}) \tag{3.4}$$

where $I =]a, b[$. We can now use the following Theorem proved by Chen and Frid in [7]:

Theorem 3.1.1. *Let Q be an open set with regular boundary ∂Q and $F \in [L^\infty(Q)]^{d+1}$ be such that $\operatorname{div}_y F$ is a bounded measure. Then there exists $F \cdot n \in L^\infty(\partial Q)$ such that for every ψ ∂Q regular deformation:*

$$\operatorname*{esslim}_{s \to 0} F(\psi(s, \cdot)) \cdot n_s(\cdot) = F \cdot n \text{ in } L^\infty(\partial Q) \ w*,$$

where n_s is a unit outward normal field of $\psi(\{s\} \times \partial Q)$.

This theorem insures that there exists $\overline{H}^\tau_\eta \cdot n \in L^\infty(]-r_\alpha, r_\alpha[^d)$ which does not depend on ψ such that

$$\overline{H}_\eta(\tilde{\psi}(s, \cdot)) \cdot n_s(\cdot) \overset{\mathcal{D}'(]-r_\alpha, r_\alpha[^d)}{\underset{s \to 0}{\longrightarrow}} \overline{H}^\tau_\eta \cdot n, \tag{3.5}$$

for every regular deformation ψ. The function n_s converges strongly in $L^1(]-r_\alpha, r_\alpha[^d)$ to n, unit outward normal field of $]-r_\alpha, r_\alpha[^d$. We notice that n does not depend on points in $]-r_\alpha, r_\alpha[^d$. So, using (3.2) and (3.3), (3.5) leads to:

$$\int_{]-r_\alpha, r_\alpha[^d} \int_I \varphi(\hat{w})\eta(\xi)a(\xi) \cdot n g^\tau_\psi(\hat{w}, \xi)\, d\xi\, d\hat{w} = \int_{]-r_\alpha, r_\alpha[^d} \overline{H}^\tau_\eta \cdot n\varphi(\hat{w})\, d\hat{w}$$

for every test function $\varphi \in \mathcal{D}(]-r_\alpha, r_\alpha[^d)$ and $\eta \in \mathcal{D}(\mathbb{R})$. The right-hand side of this equation is independent of ψ, the sequence s^j and the subsequence s^{j_p}

so $(a \cdot n)h_\psi^\tau$ also does not depend on those quantities. The result is obtained from the uniqueness of the limit. $\square$

We next prove L^1 convergence of $(a \cdot n)g^\tau$. The proof will be postponed until section 3.4.

Proposition 3.1.2. *Let I be an interval verifying $b'(\xi) = 0$ for all $\xi \in I$ and g be a χ function defined in (3.1). Then, there exists a unique trace function $(a \cdot n)g^\tau \in L^\infty(]-r_\alpha, r_\alpha[^d \times I)$ (which does not depend on the deformation ψ) such that:*

$$\operatorname*{esslim}_{s \to 0} (a \cdot n_s) g_\psi(s, \cdot, \cdot) = (a \cdot n)g^\tau \text{ in } L^1(]-r_\alpha, r_\alpha[^d \times I).$$

Notice that we can also show Proposition 3.1.2 on different hyperplanes H_j for every $j \in \{1, 2, ..., M\}$. Theorem 3.1.2 follows.

Theorem 3.1.2. *Let I be an interval verifying $b'(\xi) = 0$ for all $\xi \in I$ and f be a χ function defined by solution of (1.8) and (1.9). Then, there exists a unique trace function $(a \cdot n)f^\tau \in L^\infty(\Gamma \times I)$ (which does not depend on the deformation ψ) such that:*

$$\operatorname*{esslim}_{s \to 0} (a \cdot n_s) f_\psi(s, \cdot, \cdot) = (a \cdot n)f^\tau \text{ in } L^1_{loc}(\Gamma \times I).$$

Proof of Theorem 3.1.2: Let us consider the χ-function f associated to the solution u of (1.8) and (1.9) through Theorem 2.1.1. Then, we have: for

$\Lambda_\alpha^k \hat{z} = (0, \hat{w})$ for each $k \in \{1, 2, ..., M\}$,

$$\int_{\Sigma_\alpha^k} \int_I |A'(\xi) \cdot \hat{n_s}(\hat{z}) f_\psi(s, \hat{z}, \xi) - A'(\xi) \cdot \hat{n}(\hat{z}) f^\tau(\hat{z}, \xi)| \, d\xi d\sigma(\hat{z})$$
$$\leq \int_{]-r_\alpha, r_\alpha[^d} \int_I |a(\xi) \cdot n_s g_\psi(s, \hat{w}, \xi) - a(\xi) \cdot n g^\tau(\hat{w}, \xi)| \, d\xi \, d\sigma(\hat{w}), \tag{3.6}$$

Proposition 3.1.2 shows that $(a \cdot n_s) g_\psi(s) - (a \cdot n) g^\tau$ converges strongly in L^1 to 0 for every α. Let us fix $\epsilon > 0$, and take the collection of open susets $\{G_\epsilon^{ij}\}_{ij=1}^M$ verifying the following:

$$G_\epsilon^{ij} = \{(\hat{t}, \hat{x}) \in \Gamma | diam(\hat{x}, \overline{H}_i \cap \overline{H}_j) < \epsilon\}.$$

Finally for every compact set $K \subset\subset \Gamma$, $\{\Sigma_\alpha^k\} \bigcup \{G_\epsilon^{ij}\}$ is a covering of K with open sets of Γ so there exists an I_0 finite set such that $K \subset \bigcup_{k=1}^M \bigcup_{\alpha \in I_0} \Sigma_\alpha^k$ $\bigcup \bigcup_{i \neq j}^M \{G_\epsilon^{ij}\}$ and so

$$\int_K \int_I |A'(\xi) \cdot \hat{n_s}(\hat{z}) f_\psi(s, \hat{z}, \xi) - A'(\xi) \cdot \hat{n}(\hat{z}) f^\tau(\hat{z}, \xi)| d\xi d\sigma(\hat{z})$$
$$\leq \sum_{k=1}^M \sum_{\alpha \in I_0} \int_{\Sigma_\alpha^k} \int_I |A'(\xi) \cdot \hat{n}_s f_\psi(s, \hat{z}, \xi) - A'(\xi) \cdot \hat{n} f^\tau(\hat{z}, \xi)| d\xi d\sigma(\hat{z}) + C\epsilon.$$

where C is constant. The first part of the above second line converges to 0 when s tends to 0. Since the above inequality hold for an arbitrary positive number $\epsilon > 0$, the proof of Theorem 3.1.2 is complete. $\square$

To prove Proposition 3.1.2, we just need to prove that $(a \cdot n) g^\tau$ is reached strongly by L^1 convergence. To do this, we need to use the concept of the H-measure which is more refined than the Blow-up technique used in Vasseur [38] . So, we will follow the scheme of Panov [29], but our result extends his result.

3.2 Some properties of H-measure

We are now ready to prove the theorem which plays the key role in the proof of Proposition 3.1.2. From now on we will work with the flux A_α and the domain Q_α^k given in section 3.1. We now assume the following:

Assumption 3.2.1. *Let ν_w^r be a measure valued function on Q_α^k. Then we assume the following:*

$$\mathcal{L}_r^p = \mathrm{div}_w \int_p^\infty (A_\alpha(\lambda) - A_\alpha(p)) d\nu_w^r(\lambda)$$

is pre-compact and converges to 0 in $H_{loc}^{-1}(Q_\alpha^k)$ for all $p \in P$.

Let q be a fixed number in $E \cap P$ and let $L(p)$ be the smallest linear space $L \subset \mathbb{R}^{d+1}$ such that $Q_\alpha^k \times L$ contains the closed support *supp* μ^{pq}. We set $l(p) = dim L(p)$. Let $p_0 \in E$ be such that $l = l(p_0) = \max_{p \in E} l(p)$ and $L = L(p_0)$. Then we get the following lemma:

Lemma 3.2.1. *(Panov [27]) There is a neighborhood V of p_0 in E such that $L(p) = L$ for all $p \in V$.*

We can prove this lemma if we follow the idea given in the lemma by Panov [27] and it is a local version to the Lemma given in Panov [27]. The proof will be provided in the Appendix.

Let us denote $B_\alpha(\lambda) = (\lambda, A_\alpha(\lambda))$ and let $\pi : \mathbb{R}^{d+1} \to L$ be the orthogonal projection onto L, and set $\widetilde{B_\alpha}(\lambda) = (\pi \circ B_\alpha)(\lambda)$. We are able to see the following lemma as the main part of the proof for the result [27] with using the neighborhood V given in Lemma 3.2.1.

Proposition 3.2.1. *(Panov [27]) Assume that the assumption 3.2.1 holds. Then, there exists* $\delta > 0$ *such that* $\widetilde{B_\alpha}(p) = \widetilde{B_\alpha}(p_0)$ *for all* $p \in]p_0, p_0 + \delta[\subset E$.

From Proposition 3.2.1, we deduce the following proposition:

Proposition 3.2.2. *Assume that there exists* $\bar{p} \in E \cap P$ *such that* $\mu^{\bar{p}q} \neq 0$ *for some* $q \in E \cap P$. *Then there exists* $]p_0, p_0 + \delta[\subset E$, $\zeta \in S$ *such that* $a(p) \cdot \zeta = 0$ *for all* $p \in]p_0, p_0 + \delta[$ *and* $\mu^{pq} \neq 0$ *for all* $p \in]p_0, p_0 + \delta[\cap P$.

Proof of Proposition 3.2.2: From the assumption, l should be positive. So, we can take a non-zero ζ in L such that from Lemma 3.2.1, there exists $\delta > 0$ such that $\widetilde{B_\alpha}(p) = \widetilde{B_\alpha}(p_0)$ for all $p \in]p_0, p_0+\delta[\subset E$ and $(\zeta, B_\alpha(p)) = (\zeta, B_\alpha(p_0))$ for all $p \in]p_0, p_0 + \delta[$. Since $\mu^{p\bar{q}}$ is continuous at p_0, there exists $\delta > 0$ such that $a(p) \cdot \zeta = 0$ for all $p \in]p_0, p_0 + \delta[$ and $\mu^{pq} \neq 0$ for all $p \in]p_0, p_0 + \delta[\cap P$. The Proposition is proven. □

In [29] Panov has shown this result on the half space $\mathbb{R}^+ \times \mathbb{R}^d$ for scalar conservation laws, but we need to consider the points not verifying the non-characteristic boundary for the initial boundary value problems of scalar conservation laws, namely, $A'(u) \cdot \hat{n} = 0$. To overcome this problem, we take a H-measure excluding such points, but we will almost follow Panov's presentation.

3.3 Some convergence results

To prove Proposition 3.1.2. we will first begin by showing that the convergence holds strongly for the special deformation $\tilde{\psi}_0$ defined by:

$$\tilde{\psi}_0(s, \hat{w}) = (s, \hat{w}), \tag{3.7}$$

and then show that this holds for any deformation ψ. We have

$$g_{\psi_0}(s, \hat{w}, \xi) = g(\tilde{\psi}_0(s, \hat{w}), \xi) = g(s, \hat{w}, \xi). \tag{3.8}$$

and $\hat{m}(s, \hat{w}, \xi) = \hat{m}(\tilde{\psi}_0(s, \hat{w}), \xi)$ for all $\xi \in I$ where I an interval verifying $b'(\xi) = 0$ for all $\xi \in I$.

Before introducing the notion of a rescaled solution, let us state two lemmas. For the sake of completeness the proofs are provided in the appendix.

Lemma 3.3.1. *Let I be an interval verifying $b'(\xi) = 0$ for all $\xi \in I$. Then there exists a sequence δ_n which converges to 0 and a set $\mathcal{E} \subset]-r_\alpha, r_\alpha[^d$ with $\mathcal{L}(]-r_\alpha, r_\alpha[^d \setminus \mathcal{E}) = 0$ such that for every $\hat{w} \in \mathcal{E}$ and every $R > 0$*

$$\lim_{n\to\infty} \frac{1}{\delta_n^d} \hat{m}(]0, R\delta_n[\times(\hat{w}+]-R\delta_n, R\delta_n[^d) \times I) = 0.$$

Lemma 3.3.2. *Let I be an interval verifying $b'(\xi) = 0$ for all $\xi \in I$. Then there exists a subsequence still denoted by δ_n and a subset $\mathcal{E}'$ of $]-r_\alpha, r_\alpha[^d$ such that $\mathcal{E}' \subset \mathcal{E}$, $\mathcal{L}(]-r_\alpha, r_\alpha[^d \setminus \mathcal{E}') = 0$, and for every $\hat{w} \in \mathcal{E}'$ and every $R > 0$:*

$$\lim_{\delta_n \to 0} \int_I \int_{]-R,R[^d} |g^\tau(\hat{w}, \xi) - g^\tau(\hat{w} + \delta_n \underline{\hat{w}}, \xi)| \, d\underline{\hat{w}} \, d\xi = 0.$$

We introduce the rescaled g function defining a measure valued function. We denote

$$Q_\alpha^{k,\delta} =]0, r_\alpha/\delta[\times] - r_\alpha/\delta, r_\alpha/\delta[^d.$$

From now on we fix such a $\hat{w} \in \mathcal{E}'$. Then, we rescale the g function by introducing a new function g_δ which depends on the new variables $\underline{w} = (\underline{s}, \underline{\hat{w}}) \in Q_\alpha^{k,\delta}$ and is defined by:

$$g_\delta(\underline{s}, \underline{\hat{w}}, \xi) = g(\delta \underline{s}, \hat{w} + \delta \underline{\hat{w}}, \xi). \tag{3.9}$$

This function depends obviously on $\hat{w}$, but since it is fixed all along this section, we skip it in the notation. The function g_δ is still a χ-function and we notice that:

$$g_\delta(0, \underline{\hat{w}}, \xi) = g^\tau(\hat{w} + \delta \underline{\hat{w}}, \xi). \tag{3.10}$$

Hence we expect to gain some knowledge on $g^\tau(\hat{w}, \cdot)$ itself by studying the limit of g_δ when $\delta \to 0$. We get

$$a(\xi) \cdot \nabla_{\underline{w}} g_\delta = \partial_\xi \hat{m}_\delta, \tag{3.11}$$

where $\hat{m}_\delta$ is the nonnegative measure defined for every real $R_1^i < R_2^i$ by,

$$\hat{m}_\delta(\prod_{0 \le i \le d} [R_1^i, R_2^i] \times I) = \frac{1}{\delta^d} \hat{m}(\prod_{0 \le i \le d} [y_i + \delta R_1^i, y_i + \delta R_2^i] \times I), \tag{3.12}$$

for the intervals I verifying $b'(\xi) = 0$ for all $\xi \in I$. We now pass to the limit when δ goes to 0 in the scaling, which leads to the following proposition.

Proposition 3.3.1. *(Kwon and Vasseur [19]) Let I be an interval verifying $b'(\xi) = 0$ for all $\xi \in I$. Then there exist a sequence δ_n which goes to 0,* *and a χ-function $g_\infty \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d \times I)$ such that g_{δ_n} converges to g_∞ in $L^\infty_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}^d \times I)w\star$ and,*

$$\begin{aligned} &a(\xi) \cdot \nabla_{\underline{w}} g_\infty = 0, \\ &a(\xi) \cdot n g_\infty(0, \underline{\hat{w}}, \xi) = a(\xi) \cdot n g^\tau(\hat{w}, \xi) \end{aligned} \tag{3.13}$$

for almost every $(\underline{\hat{w}}, \xi) \in \mathbb{R}^d \times I$.

For the convergence of the second part in (3.13), we used the fact that we have uniform convergence in a weak topology with respect to variable s and so the weak limit of the variable s is equal to the work of limit (see [19]). Thus, from (3.13), we deduce that,

$$a(\xi) \cdot n g_\infty(\underline{w}, \xi) = a(\xi) \cdot n g^\tau(\hat{w}, \xi) \tag{3.14}$$

for $\xi \in I$. Indeed, we get (3.14) from the transport equation in (3.13).

We consider a sequence δ_n given in Lemma 3.3.1 and Lemma 3.3.2 and from now on, set $g_{\delta_n} = g_n$ We are able to show the following equivalent Lemma.

Lemma 3.3.3. *Let J be an subinterval of I where I is an interval verifying $b'(\xi) = 0$ for all $\xi \in I$ and consider g a χ-function verifying (3.1). Then the following are equivalent:*

- $\operatorname{ess\,lim}_{s\to 0} a \cdot n_s g(s, \cdot, \cdot) = a \cdot n g^\tau$ *in* $L^1_{loc}(]-r_\alpha, r_\alpha[^d \times J)$
- $a(\xi) \cdot n g_n(\underline{w}, \xi)$ *converges in* $L^1_{loc}(]-r_\alpha, r_\alpha[^{d+1} \times J)$ *for a.e w.*

Proof of Lemma 3.3.3: Assume that $\operatorname{ess\,lim}_{s\to 0} a \cdot n_s g(s, \cdot, \cdot) = a \cdot n g^\tau$ in $L^1_{loc}(]-r_\alpha, r_\alpha[^d \times J)$. Let $\rho \in C^\infty_c(]-r_\alpha, r_\alpha[^d)$ and $\Phi \in C^\infty_c(]-r_\alpha, r_\alpha[^{d+1} \times J)$. Applying the change of variables and assumption provides the following inequality:

$$\begin{aligned}&\int_J\int |a(\xi)\cdot n||g(w+\delta_n\underline{w},\xi)-g^\tau(\hat{w}+\delta_n\underline{\hat{w}},\xi)||\rho(\hat{w})|d\hat{w}d\xi,\\ =&\int_J\int |a(\xi)\cdot n||g(\underline{s_0}\delta_n,\hat{z},\xi)-g^\tau(\hat{z},\xi)||\rho(\hat{z}-\delta_n\underline{\hat{w}})|d\hat{z}d\xi,\end{aligned} \tag{3.15}$$

converges to 0 as n tends to ∞. Let us next integrate (3.15) with respect to $\underline{w}$, which implies that

$$\int_J\int\int |a(\xi)\cdot n||g(w+\delta_n\underline{w},\xi)-g^\tau(\hat{w}+\delta_n\underline{\hat{w}},\xi)||\Phi(\underline{w},\xi)||\rho(\hat{w})|d\underline{w}d\hat{w}d\xi,$$

goes to 0 whenever $n\longrightarrow\infty$. Therefore, after subtracting a subsequence, we deduce $a(\xi)\cdot ng_n(\underline{w},\xi)$ converges strongly in $L^1_{loc}(]-r_\alpha,r_\alpha[^{d+1}\times J)$.

Conversely, assume that $a(\xi)\cdot ng_n(\underline{w},\xi)$ converges strongly in $L^1_{loc}(]-r_\alpha,r_\alpha[^{d+1}\times J)$. From (3.14), We see that for almost every w,

$$a(\xi)\cdot ng_n(\underline{w},\xi)\longrightarrow a(\xi)\cdot ng^\tau(\hat{w},\xi) \tag{3.16}$$

in $L^1_{loc}(]-r_\alpha,r_\alpha[^{d+1}\times J)$. Then using the change of variables and Fubini's theorem, we obtain, for $\Phi_1\in C_c^\infty(]-r_\alpha,r_\alpha[^d)$ and $\Phi_2\in C_c^\infty(]-r_\alpha,r_\alpha[^{d+1}\times J)$,

$$\begin{aligned}&\int_J\int\int |a(\xi)\cdot n||g_n(\underline{w},\xi)-g_n^\tau(\hat{w},\xi)||\Phi_2(\underline{w},\xi)\Phi_1(\hat{w})|d\underline{w}d\hat{w}d\xi,\\ \leq&\int_J\int\int |a(\xi)\cdot n||g(\underline{s_0}\delta_n,\hat{z},\xi)-g^\tau(\hat{z},\xi)||\Phi_1(\hat{z}-\delta_n\underline{\hat{w}})\Phi_2(\underline{w},\xi)|\\ &\qquad d\hat{z}d\underline{w}d\xi.\end{aligned}$$

Thus, we get the following inequality.

$$\begin{aligned}
&\int_J\int\int |a(\xi)\cdot n||g(\underline{s_0}\delta_n,\hat{z},\xi)-g^\tau(\hat{z},\xi)||\Phi_1(\hat{z})||\Phi_2(\xi)|d\hat{z}d\underline{w}d\xi \\
&\le \int_J\int\int |a(\xi)\cdot n||g(\underline{s_0}\delta_n,\hat{z},\xi)-g^\tau(\hat{z},\xi)||\Phi_1(\hat{z}-\delta_n\underline{w}) \\
&\qquad -\Phi_1(\hat{z})||\Phi_2(\underline{w},\xi)|d\hat{z}d\underline{w}d\xi, \\
&+\int_J\int\int |a(\xi)\cdot n||g(\underline{s_0}\delta_n,\hat{z},\xi)-g^\tau(\hat{z},\xi)||\Phi_1(\hat{z}-\delta_n\underline{w})| \\
&\qquad |\Phi_2(\underline{w},\xi)|d\hat{z}d\underline{w}d\xi,
\end{aligned} \tag{3.17}$$

converges to 0 by (3.16). It follows that there exists a subsequence $\{\delta_r\}_{r=1}^\infty$ of $\{\delta_n\}_{n=1}^\infty$ such that

$$a\cdot ng(\delta_r\underline{s_0},\cdot,\cdot)\longrightarrow a\cdot ng^\tau \ in \ L^1_{loc}(]-r_\alpha,r_\alpha[^d\times J).$$

for almost every $\underline{s_0}$. Since $s\longmapsto a\cdot n_s$ is smooth and Proposition 3.1.1 holds, the proof is complete. □

3.4 Proof of Proposition 3.1.2

We now prove Proposition 3.1.2. From now on we will use the following measure valued function,

$$\nu^r_w(\xi)=\delta(\xi-v^r_\alpha(w)), \tag{3.18}$$

where I is an interval verifying $b'(\xi)=0$ for all $\xi\in I$, and we notice that $v^r_\alpha(w)=\int_I g_r(w,\xi)d\xi-b=\max\{a,\min\{u_\alpha,b\}\}$ for $I=]a,b[$. Then, we obtain the following lemma.

Lemma 3.4.1. *Let $E=\{p|a(p)\cdot n\neq 0\}\ \cap\ I$. Consider an interval $J\subset I$. Then the following are equivalent:*

- *There exists a H-measure μ^{pq} corresponding to (3.18) such that $\mu^{pq} = 0$ for all $p, q \in P \cap E \cap J$.*

- *There exist a subsequence $\{\delta_n\}$ of $\{\delta_r\}$ such that $a(\xi) \cdot n g_n(\underline{w}, \xi)$ converges in $L^1_{loc}(]-r_\alpha, r_\alpha[^{d+1} \times J)$ for almost every w.*

Proof of Lemma 3.4.1: Suppose that $\mu^{pq} = 0$ for all $p, q \in P \cap E \cap J$. In particular, $\mu^{pp} = 0$ for all $p \in P \cap E \cap J$. Using the definition of the H-measure, we have that: $\forall \theta \in C^\infty_{loc}(\mathbb{R}^d)$, there exists a subsequence $\{\delta_n\}$ of $\{\delta_r\}$ such that

$$\lim_{n\to\infty} \int_{\mathbb{R}^{d+1}} |\theta(\underline{w})|^2 |U^p_n(\underline{w})|^2 d\underline{w} = 0,$$

which implies that $V_n(\underline{w}, p)$ converges strongly in $L^1_{loc}(\mathbb{R}^{d+1})$ for all $p \in P \cap E \cap J$. It remains to show that $a(\xi) \cdot n g_n(\underline{w}, \xi)$ converges in $L^1_{loc}(]-r_\alpha, r_\alpha[^{d+1} \times J)$ for almost every w. To do this, we need to understand the relation between V_r and the χ function g. Actually, from the definition of the measure valued function ν^r_w given in (3.18), we are able to see that

$$g_n(\underline{w}, p) = \begin{cases} V_n(\underline{w}, p) & \text{if } p \in [0, \infty) \cap I \\ V_n(\underline{w}, p) - 1 & \text{if } p \in (-\infty, 0[\cap I. \end{cases} \tag{3.19}$$

Since

$$\forall p \in E^c, \quad a(p) \cdot n = 0, \tag{3.20}$$

we see that $a(p) \cdot n g_n(\underline{w}, p)$ converges in $L^1_{loc}(]-r_\alpha, r_\alpha[^{d+1} \times J)$ for almost every w.

Let us next prove the converse direction. Assume that $a(p) \cdot n g_n(\underline{w}, p)$ converges in $L^1_{loc}(]-r_\alpha, r_\alpha[^{d+1} \times J)$ for almost every w. Since if $p \in E$, $a(p) \cdot n \neq$

0. we can take a subsequence $\{\delta_m\}_{m=1}^{\infty}$ of $\{\delta_n\}_{n=1}^{\infty}$ such that

$$g_m(\underline{w}, p) \longrightarrow g^{\tau}(\hat{w}, p) \tag{3.21}$$

in $L^1_{loc}(]-r_\alpha, r_\alpha[^{d+1})$ for almost every $p \in P \cap E \cap J$. We can now construct a H-measure with using the measure value function ν_w^m corresponding to g_m. Thus, from Theorem 2.2.1, there is a subsequence $\{U_k\}$ of $\{U_m\} = \{U_m^p\}_{p \in P}$ and we can define H-measure μ^{pq} with respect to the ν_w^m. To complete the proof, we need to show that

$$U_k^p(\underline{w}) \longrightarrow 0$$

in $L^2_{loc}(\mathbb{R}^{d+1})$ for any $p \in E \cap P \cap J$. Actually, using two results (3.19) and (3.21), we deduce that

$$U_k^p(\underline{w}) \longrightarrow 0$$

in $L^1_{loc}(\mathbb{R}^{d+1})$ for any $p \in E \cap P \cap J$. It follows that

$$U_k^p(\underline{w}) \longrightarrow 0 \tag{3.22}$$

in $L^2_{loc}(\mathbb{R}^{d+1})$ for any $p \in E \cap P \cap J$. Indeed,

$$\int_{]-r_\alpha, r_\alpha[^{d+1}} |U_k^p(\underline{w})|^2 d\underline{w} \leq 3 \int_{]-r_\alpha, r_\alpha[^{d+1}} |U_k^p(\underline{w})| d\underline{w} \longrightarrow 0.$$

Combining (3.22) and the Lebegue dominated convergence theorem gives

$$\mu^{pq} = 0.$$

for all $p, q \in E \cap P \cap J$. The proof is complete. $\square$

Proof of Proposition 3.1.2: Let us first prove Proposition 3.1.2 for the special deformation ψ_0 defined in (3.7). We prove our result using mathematical induction in dimension $d \geq 1$. When $d = 1$, we have shown the result in Kwon and Vasseur [19]. Assume that Proposition 3.1.2 holds for $d - 1$.

Let us denote a convenient notation.

$$\mathcal{B} = \{]\alpha, \beta[\subset I | a(\xi) \cdot \zeta = 0 \ for \ all \ \xi \in]\alpha, \beta[\ for \ some \ \zeta \neq 0\} \tag{3.23}$$

and ζ by $\zeta = (\zeta^0, \hat{\zeta})$. Let $]\alpha, \beta[\in \mathcal{B}$, we can take a nonzero ζ such that $a(\xi) \cdot \zeta = 0$ for all $\xi \in]\alpha, \beta[$. Thus, we are able to find a new coordinate by the following linear map as mentioned in Panov [29]:

$$\Phi(w) = z(s, \hat{w}) = (s, sC + A\hat{w})$$

where A is a nonsingular matrix, C is a constant vector, and $z_d = w \cdot \zeta$. Then we obtain the new representation of a:

$$\tilde{a}(\xi) = \Phi(a(\xi)).$$

The following lemma is devoted to reducing the dimension.

Lemma 3.4.2. *(Panov [29]) Consider a χ function g given in (3.1). Then, there exists a nonnegative finite measure $\tilde{m}$ such that*

$$\tilde{a}(\xi) \cdot \nabla_z g = |detA| \partial_\xi \tilde{m} \tag{3.24}$$

in $\mathcal{D}'$ where $\tilde{m}(x, \xi) = \hat{m}(\Phi^{-1}(z, \xi))$ for any $\xi \in]\alpha, \beta[$.

Indeed,

$$\tilde{a}_d(\xi) = a(\xi) \cdot \zeta = 0 \tag{3.25}$$

Denote $z = (\hat{z}, z_d)$ and $\tilde{a} = (\check{a}, \tilde{a}_d)$. Then, from (3.25), (3.24) can be represented by :

$$\check{a}(\xi) \cdot \nabla_{\hat{z}} g(\hat{z}, z_d) = |detA| \partial_\xi \tilde{m}(\hat{z}, z_d), \tag{3.26}$$

for $a.e$ z_d which implies that (1.4) and (3.26) for solution v_α verifies the assumption of the case of dimension $d-1$. The proof of the existence of a locally finite measure is provided in Panov [29]. Thus,

$$\operatorname*{esslim}_{s\to 0} \check{a} \cdot \check{n}_s g_{\psi_0}(s, z_d) = \check{a} \cdot \check{n} g^\tau(z_d) \ in \ L^1_{loc} \tag{3.27}$$

for $a.e$ z_d where $n = (\check{n}, \check{n}_d)$ is the unit outward normal to $]-r_\alpha, r_\alpha[^d$. Indeed, if ζ is parallel to n, then $a(\xi) \cdot n = 0$, which deduces

$$\check{a}(\xi) \cdot \check{n} = -a_0(\xi) = a(\xi) \cdot n = 0.$$

Applying the Lebegue Dominated convergence to (3.27), then, thanks to the boundedness of g, a, the domain, and using a change of variables, (3.27) implies that there exists a subsequence $\{s_m\}$ such that

$$\operatorname*{esslim}_{s_m\to 0} a \cdot n_{s_m} g_{\psi_0}(s_m) = a \cdot n g^\tau \ in \ L^1_{loc}(]-r_\alpha, r_\alpha[^d \times]\alpha, \beta[) \tag{3.28}$$

for every $]\alpha, \beta[\in \mathcal{B}$. We can also show that (3.28) does not depend on any sequence $\{s_m\}$ by Proposition 3.1.1.

We may assume the collection $\mathcal{B}$ to be countable. So, we can take a sequence $\{\delta_r\}$ which is independently of $]\alpha, \beta[\in \mathcal{B}$ by using the diagonal extraction. Thus, from Lemma 3.3.3 and Lemma 3.4.1, (3.28) deduces

$$\mu^{pq} = 0 \tag{3.29}$$

for every $p, q \in E \bigcap P \bigcap]\alpha, \beta[$ where $]\alpha, \beta[\in \mathcal{B}$.

We next show that the assumption (3.2.1) holds for any $p \in \mathbb{R}$. A distribution form (3.2.1), $\mathcal{L}_r^p$, is the same as the following:

$$\begin{aligned}
&\mathcal{L}_r^p = \mathrm{div}_w[sign^+(v_\alpha^r(w) - p)(A_\alpha(v_\alpha^r(w)) - A_\alpha(p))] \\
&= \mathrm{div}_w \int_p^b a(\xi) g_r(w, \xi) d\xi \\
&= \int_p^b \partial_\xi \hat{m}_r d\xi \\
&= \hat{m}_r(w, b) - \hat{m}_r(w, p),
\end{aligned}$$

which converges to 0 in $\mathcal{M}(]0, r_\alpha[\times] - r_\alpha, r_\alpha[^d)$ for almost every p, b by the following fact.

$$\int_{]0, r_\alpha[\times]-r_\alpha, r_\alpha[^d} \int_I |\hat{m}_r(w, \xi)| d\xi dw$$

converges to 0 as r goes to ∞ by the Lemma 3.3.1. Thus, up to subsequence, $\hat{m}_r(\cdot, \xi)$ converges to 0 when r tends to ∞ in $\mathcal{M}(]0, r_\alpha[\times] - r_\alpha, r_\alpha[^d)$ for almost every $\xi \in I$. We can also see that $\mathcal{L}_r^p$ is equicontinous for p in H_{loc}^{-1} by Panov' result. Thus, assumption (3.2.1) holds for any $p \in \mathbb{R}$.

Assume that there exists $\bar{p} \in E \cap P$ such that $\mu^{\bar{p}q} \neq 0$ for some $q \in E \cap P$. Then, Proposition 3.2.2 implies that there exists an interval $]\alpha, \beta[\subset E$ such that $a(\xi) \cdot \eta = 0$ for all $\xi \in]\alpha, \beta[$ for some $\eta \neq 0$ and $\mu^{pq} \neq 0$ for all $p \in]\alpha, \beta[\cap P$. Then, from(3.29), we can conclude the contradiction. Indeed, $\mu^{pq} \neq 0$ for every

$p, q \in E \bigcap P \bigcap]\alpha, \beta[$. We therefore get

$$\mu^{pq} = 0$$

for all $p, q \in E \cap P$. Since $s \longmapsto a \cdot n_s$ is smooth, Lemma 3.3.3 and Lemma 3.4.1 yield

$$\operatorname*{esslim}_{s \to 0} a \cdot n_s g_{\psi_0}(s, \cdot, \cdot) = a \cdot n g^\tau \tag{3.30}$$

in $L^1_{loc}(] - r_\alpha, r_\alpha[^d \times I)$, which holds for any deformation ψ thanks to Lemma 3.4.3. $\square$

Lemma 3.4.3. *Let $J =]a, b[$ be an interval such that $-L \leq a < b \leq L$, and $g \in L^\infty(]0, 1[\times] - r_\alpha, r_\alpha[^d \times J)$ be a χ-function. We consider $\beta \in C^0(]0, 1[\times] - r_\alpha, r_\alpha[^d \times J)$ and denote $\beta_0 = \beta(0, \cdot, \cdot)$. We assume that there exists $\beta_0 g^\tau \in L^\infty(] - r_\alpha, r_\alpha[^d \times J)$ such that $\beta(s)g(s)$ converges weakly to $\beta_0 g^\tau$. Then the following two propositions are equivalent:*

- *$\beta(s)g(s)$ converges strongly to $\beta_0 g^\tau$ in $L^1_{\mathrm{loc}}(] - r_\alpha, r_\alpha[^d \times J)$ when $s \to 0$,*
- *For almost every $(\hat{w}, \xi) \in] - r_\alpha, r_\alpha[^d \times J$, $\beta_0(\hat{w}, \xi)g^\tau(\hat{w}, \xi)$ is equal to $\mathrm{sign}(\xi)\beta_0(\hat{w}, \xi)$ or 0.*

The proof of lemma 3.4.3 is provided in the Appendix.

We have finished proving the existence of strong trace and L^1 convergence for $b'(\xi) = 0$. Next let us show this result for the parabolic-hyperbolic case.

Chapter 4

The general case of parabolic-hyperbolic type

In this chapter we will deal with the general parabolic-hyperbolic type. We first need to show the L^1 convergence Proposition 4.2.1 with the well-known trace result, Theorem 4.1.1 in Lions and Magenes [20], and finally we give the general proof of Theorem 1.4.2.

4.1 Trace theorem

We now assume that there exists an interval J and $c > 0$ such that

$$b'(\xi) > c \tag{4.1}$$

for all $\xi \in J$. Here in order to show the existence of the strong trace and L^1 convergence, we introduce a useful trace theorem mentioned in [20]. Furthermore, we need the regularity on Γ assumed in Lions and Magenes (1.10) to use the trace theorem. Let us denote a special space: for a fixed $\epsilon > 0$,

$$W(]0,\epsilon[) = \{u \in L^2(0,\epsilon; L^2(\Gamma)) | \partial_s u \in L^2(0,\epsilon; L^2(\Gamma))\}$$

Theorem 4.1.1. *(Lions and Magenes [20]) Assume that $u \in W(]0,\epsilon[)$. Then $u \in C([0,\epsilon]; L^2(\Gamma))$.*

4.2 L^1 convergence on the non-degenerate range

We prove the following result:

Proposition 4.2.1. *Let f be a χ function verifying (2.6) with assumption (4.1). then there exists $f^\tau \in L^\infty(\Gamma \times J)$ such that any regular deformation ψ,*

$$\operatorname*{esslim}_{s \to 0} \int_K \int_J |f(\psi(s,\hat{z}),\xi) - f^\tau(\hat{z},\xi)| d\xi d\sigma(\hat{z}) = 0.$$

for each $K \subset\subset \Gamma$ where $d\sigma$ is the volume element of Γ.

Proof of Proposition 4.2.1: Let us consider the function:

$$s \longmapsto b(u)(\psi(s,\hat{z}))$$

for a fixed $\hat{z} = (\hat{t},\hat{x}) \in \Gamma$. Taking the derivative with respect to s, we obtain:

$$\partial_s b(u)(\psi(s,\hat{z})) = \partial_s b(u)(\hat{t},\hat{\psi}(s,\hat{x})) = \nabla_x b(u) \cdot \partial_s \hat{\psi}(s,\hat{x}). \tag{4.2}$$

Using (1.10), (4.2), and the Fubini theorem , we deduce:

$$b(u)(\psi(s,\cdot)) \in W(]0,\epsilon[).$$

Indeed,

$$\begin{aligned}
&\int_0^\epsilon \int_\Gamma |\partial_s b(u)(\psi(s,\hat{z}))|^2 d\sigma(\hat{z}) ds \\
&\leq \int_0^\epsilon \int_\Gamma |\nabla_x b(u)(\psi(s,\hat{z}))|^2 |\partial_s \hat{\psi}(s,\hat{x})|^2 d\sigma(\hat{z}) ds, \\
&\leq \| \partial_s \hat{\psi} \|_{L^\infty(\bar{Q})} \int_0^T \int_{\Omega_\epsilon} |\nabla_x b(u)(t,x)|^2 dx dt \\
&\leq M \int_0^T \int_\Omega |\nabla_x b(u)(t,x)|^2 dx dt
\end{aligned}$$

where $\Omega_\epsilon = \{\hat{\psi}(s,x) | 0 \le s \le \epsilon, \quad x \in \partial\Omega\}$ and $M = M(\hat{\psi})$, and

$$\begin{aligned}\int_0^\epsilon \int_\Gamma |b(u)(\psi(s,\hat{z}))|^2 d\sigma(\hat{z}) ds &\le \int_0^T \int_{\Omega_\epsilon} |b(u)(t,x)|^2 dx dt, \\ &\le \int_0^T \int_\Omega |b(u)(t,x)|^2 dx dt < \infty.\end{aligned}$$

Thus, Theorem 4.1.1 gives the following property:

$$b(u)(\psi(s,\cdot)) \in C([0,\epsilon]; L^2(\Gamma)).$$

Let us define a χ function by

$$f^\tau(\hat{z},\xi) = \chi(b^{-1}(b(u))^\tau(\hat{z})), \xi)$$

for any $\xi \in J$. Notice that b is one to one on J. Therefore, by the following inequality, we prove L^1 convergence.

$$\begin{aligned}&\int_\Gamma \int_J |f(\psi(s,\hat{z}),\xi) - f^\tau(\hat{z},\xi)| d\xi d\sigma(\hat{z}), \\ &\le \frac{1}{c} \int_\Gamma \int_{b(J)} |f(\psi(s,\hat{z}),\zeta) - f^\tau(\hat{z},\zeta)| d\zeta d\sigma(\hat{z}), \\ &\le \frac{1}{c} \int_\Gamma |b(u)(\psi(s,\hat{z})) - [b(u)]^\tau(\hat{z})| d\sigma(\hat{z}) \longrightarrow 0.\end{aligned}$$

as $s \to 0$. The proof is complete. □

We now consider the general case of $b'(\xi)$ in the next section.

4.3 The proof of Theorem 1.4.2

In this section we use a general method in order to show the main result, Theorem 1.4.2, such that we have to work in the general case of $b'(\xi)$

on the whole interval $]-L,L[$. We first need to show that the weak trace of $g_\psi(\cdot,\cdot,\xi)$ is uniquely defined at every ξ verifying $b'(\xi)\neq 0$ and the weak trace of $a\cdot n_s g_\psi(\cdot,\cdot,\xi)$ is also uniquely defined at every ξ verifying $b'(\xi)=0$ for every Γ regular deformation ψ and then we prove the main result.

To do this, we mention the following lemma and the proof will be provided in the Appendix.

Lemma 4.3.1. *(Kwon and Vasseur [19]) Let $\mathcal{O}$ be an open set of $\mathbb{R}^N$, $I=]a,b[$ be an interval such that $-L\leq a<b\leq L$, and $f_n\in L^\infty(\mathcal{O}\times I)$ be a sequence of χ-functions converging weakly to $f\in L^\infty(\mathcal{O}\times I)$. We denote $u_n(\cdot)=\int_I f_n(\cdot,\xi)\,d\xi$ and $u(\cdot)=\int_I f(\cdot,\xi)\,d\xi$. Then for almost every $z\in\mathcal{O}$, the function $f(z,\cdot)$ lies in $BV(I)$. Moreover, the three following propositions are equivalent:*

- *f_n converges strongly to f in $L^1_{\mathrm{loc}}(\mathcal{O}\times I)$,*
- *u_n converges strongly to u in $L^1_{\mathrm{loc}}(\mathcal{O})$,*
- *f is a χ-function.*

Since $\|g_\psi(s,\cdot,\cdot)\|_{L^\infty}\leq 1$, there exists $\{s^{j_p}\}$ and $h^\tau_\psi\in L^\infty(\Gamma\times]-L,L[)$ such that

$$g_\psi(s^{j_p},\cdot,\cdot)\overset{H^{-1}\cap L^\infty W*}{\longrightarrow} h^\tau_\psi \qquad \text{when } j_p\to+\infty. \tag{4.3}$$

From Lemma 4.3.1 and (4.3), $h^\tau_\psi(\hat{w},\cdot)$ is a BV function, so it is continuous almost everywhere. Thus, it is sufficient to work at continuity points. Let us consider a continuity point ξ. We divide the point $\xi\in]-L,L[$ into three cases:

1. There exists an open interval I_ξ such that $b'(\zeta) = 0$ for all $\zeta \in I_\xi$.

2. There exists an sequence $\{\xi_j\}$ converging to ξ such that $b'(\xi) = 0$ and $b'(\xi_j) \neq 0$.

3. There exists an open interval J_ξ such that $b'(\zeta) \neq 0$ for all $\zeta \in J_\xi$.

For the first case, by Proposition 3.1.1, $a \cdot n_s h_\psi^\tau(\cdot, \xi)$ is uniquely defined and for the third case, we can also show that $h_\psi^\tau(\cdot, \xi)$ is uniquely determined by Proposition 4.2.1. Finally, it remains to show this argument for the second case. Let us fix a $\hat{w} \in]-r_\alpha, r_\alpha[^d$. Since $h_\psi^\tau(\hat{w}, \xi_j)$ pointwise converges to $h_\psi^\tau(\hat{w}, \xi)$ and $h_\psi^\tau(\hat{w}, \xi_j)$ is uniquely defined for each ξ_j, $h_\psi^\tau(\hat{w}, \xi)$ is also defined uniquely. Let us denote $a \cdot nh_\psi^\tau(\cdot, \cdot, \xi)$ by $a \cdot ng^\tau(\cdot, \cdot, \xi)$ at every ξ verifying $b'(\xi) = 0$ and $h_\psi^\tau(\cdot, \cdot, \xi)$ by $g^\tau(\cdot, \cdot, \xi)$ at every ξ verifying $b'(\xi) \neq 0$.

Next we prove the main result. We introduce a definition in order to apply Lemma 3.4.3.

Definition 4.3.1. *Let $(\hat{w}, \xi) \in]-r_\alpha, r_\alpha[^d$ be a fixed point. We say that a function g^τ is suitable at a point $(\hat{w}, \xi)$ if the following holds:*

- *$a(\xi) \cdot ng^\tau(\hat{w}, \xi)$ is equal to $\mathrm{sign}(\xi)a(\xi) \cdot n$ or 0 at all ξ verifying $b'(\xi) = 0$,*

- *$g^\tau(\hat{w}, \xi)$ is -1 or 0 or 1 at every ξ verifying $b'(\xi) \neq 0$.*

Notice that χ-functions are suitable at almost every point $(\hat{w}, \xi) \in]-r_\alpha, r_\alpha[^d \times]-L, L[$. We will consider three cases.

(i) Let us consider $\mathcal{A}_1$ the set of maximal open intervals such that $b'(\xi) = 0$. Notice that those intervals are disjoints so:

$$\sum_{I \in \mathcal{A}_1} |I| \leq 2r_\alpha.$$

Hence $\mathcal{A}_1$ is at most countable. We want now to construct a countable covering of intervals of the set $\{\xi | b'(\xi) \neq 0\}$. Notice that

$$\{\xi | b'(\xi) \neq 0\} = \cup_{n=1}^{\infty} \{\xi \mid |b'(\xi)| \geq 1/n\} = \cup_{n=1}^{\infty} B_n.$$

For each $\xi \in B_n$, there exists an interval I_ξ such that b' is far from zero globally on I_ξ. Since $B_n \subset \cup_{\xi \in B_n} I_\xi$ and B_n is compact, there is a finite number of I_ξ covering B_n. Let us denote $\mathcal{A}_2$ the union of those covering for all n. This gives a countable covering of intervals of the set $\{\xi | b'(\xi) \neq 0\}$.

(ii) For any interval I belonging to $\mathcal{A}_1$, Proposition 3.1.2 and Lemma 3.4.3 ensure that g^τ is suitable at almost every point $(\hat{w}, \xi) \in]-r_\alpha, r_\alpha[^d \times I$. We want to prove the same property for any interval $I \in \mathcal{A}_2$. Consider such an interval. From Proposition 4.2.1, we can easily show that g^τ is suitable at almost every $(\hat{w}, \xi) \in]-r_\alpha, r_\alpha[^d \times I$ for every $I \in \mathcal{A}_2$. To sum up, we have shown that for any $I \in \mathcal{A}_1 \cup \mathcal{A}_2$, the function g^τ is suitable for almost every $(\hat{w}, \xi) \in]-r_\alpha, r_\alpha[^d \times I$.

(iii) Since $\mathcal{A}_1 \cup \mathcal{A}_2$ is countable, we have that for almost every $\hat{w} \in]-r_\alpha, r_\alpha[^d$, g^τ is suitable almost everywhere in $\{\hat{w}\} \times I$, for **any** $I \in \mathcal{A}_1 \cup \mathcal{A}_2$. Let us fix such a $\hat{w}$. From Lemma 4.3.1, $g^\tau(\hat{w}, \cdot)$ is a BV function, so it is continuous

almost everywhere. Let us consider a continuity point ξ. If $b'(\xi) \neq 0$, then there exists $I \in \mathcal{A}_2$ such that $\xi \in I$. So from (ii), g^τ is suitable at this point $(\hat{w}, \xi)$. The same conclusion holds if $\xi \in I$ with $I \in \mathcal{A}_1$. The last situation corresponds to a $\xi \in]-L, L[$ verifying $b'(\xi) = 0$ but for which there exists a sequence ξ_n converging to ξ and verifying $b'(\xi_n) \neq 0$ for every n. But for all those ξ_n, $g^\tau(\hat{w}, \cdot)$ is a χ-function function on a neighborhood of ξ_n so g^τ is suitable at $(\hat{w}, \xi_n)$. Since $g^\tau(\hat{w}, \cdot)$ is continuous at ξ, $a(\cdot) \cdot n g^\tau(\hat{w}, \cdot)$ is also continuous at this point. The limit $a(\xi) \cdot n g^\tau(\hat{w}, \xi)$ can only be $\mathrm{sign}(\xi) a(\xi) \cdot n$ or 0. This implies that $g^\tau(\hat{w}, \xi)$ is suitable at $(\hat{w}, \xi)$. We have shown the property for almost every $(\hat{w}, \xi) \in]-r_\alpha, r_\alpha[^d \times]-L, L[$. Thus, from Lemma 3.4.3. we conclude the following: for every deformation ψ,

$$\operatorname*{esslim}_{s \to 0} \int_{]-r_\alpha, r_\alpha[^d} \int_{-L}^{L} |a(\xi) \cdot n_s g_\psi(s, \hat{w}, \xi) - a(\xi) \cdot n g^\tau(\hat{w}, \xi)| \mathbb{I}_{\{b'(\xi)=0\}} \\ + |g_\psi(s, \hat{w}, \xi) - g^\tau(\hat{w}, \xi)| |\mathbb{I}_{\{b'(\xi)>0\}}| d\xi d\sigma(\hat{w}) = 0. \tag{4.4}$$

Let us define $[G_h u]^\tau(\hat{z})$ by the following:

$$[G_h u]^\tau(\hat{z}) = \int_{-L}^{L} A'(\xi) \cdot \hat{n}(\hat{z}) f^\tau(\hat{z}, \xi) k(\xi) \mathbb{I}_{\{b'(\xi)=0\}} + f^\tau(\hat{z}) h_1(\hat{z}, \xi) \mathbb{I}_{\{b'(\xi)>0\}} d\xi$$

for $k \in L^\infty(\mathbb{R})$ and $h_1 \in L^\infty(\Gamma \times \mathbb{R})$. We notice that $A'(\xi) \cdot n(\hat{z}) f^\tau(\hat{z}, \xi)$ for ξ verifying $b'(\xi) = 0$ and $f^\tau(\hat{z}, \xi)$ for ξ verifying $b'(\xi) > 0$ are defined in $L^\infty(\Gamma \times]-L, L[)$ respectively in Theorem 3.1.2 and Proposition 4.2.1. Let us fix $\epsilon > 0$ and us consider $\{G_\epsilon^{ij}\}_{ij=1}^M$ in the proof of Theorem 3.1.2. Then, we have: for $\Lambda_\alpha^k \hat{z} = (0, \hat{w})$ for each $k \in \{1, 2, ..., M\}$,

$$\begin{aligned}&\int_{\Sigma_\alpha^k} |G_h u(\psi(s, \hat{z})) - [G_h u]^\tau(\hat{z})| d\sigma(\hat{z}) \\ &\qquad \leq \int_{]-r_\alpha, r_\alpha[^d} \int_{-L}^{L} |a(\xi) \cdot n_s g_\psi(s, \hat{w}, \xi) - a(\xi) \cdot n g^\tau(\hat{w}, \xi)||k(\xi)| \mathbb{I}_{\{b'(\xi)=0\}} \\ &\qquad\qquad + |g_\psi(s, \hat{w}, \xi) - g^\tau(\hat{w}, \xi)||h_1(\hat{w}, \xi)| \mathbb{I}_{\{b'(\xi)>0\}} d\xi d\sigma(\hat{z}).\end{aligned} \tag{4.5}$$

Thus, for every compact set $K \subset\subset \Gamma$, there exists an I_0 finite set such that $K \subset \bigcup_{k=1}^{M} \bigcup_{\alpha \in I_0} \Sigma_\alpha^k \bigcup \bigcup_{i \neq j}^{M} \{G_\epsilon^{ij}\}$ and we have

$$\begin{aligned}&\int_K |G_h u(\psi(s, \hat{z})) - [G_h u]^\tau(\hat{z})| d\sigma(\hat{z}) \\ &\qquad \leq \sum_{\alpha \in I_0} \sum_{k=1}^{M} \int_{\Sigma_\alpha^k} |G_h u(\psi(s, \hat{z})) - [G_h u]^\tau(\hat{z})| d\sigma(\hat{z}) + C\epsilon.\end{aligned} \tag{4.6}$$

where C is constant. The first part of the above second line converges to 0 when s tends to 0 thanks to (4.5). Since the above inequality (4.6) holds for arbitrary positive number $\epsilon > 0$, the proof of Theorem 1.4.2 is complete. $\square$

Chapter 5

Applications

This chapter shows why our trace result is a powerful tool for the uniqueness proof of scalar conservation laws. We here deal with more general result by Panov for scalar conser vation laws and we will show a simple uniqueness proof of (1.2) and (1.4) as an application of Theorem 1.4.2. Here we will follow Perthame's presentation given in [30]. The results of uniqueness obtained through different approaches appear in many articles. For more references, see Ben Moussa, Szepessy [3] and Imbert, Vovelle [17]. As a similar result, recently in [4], Bürger, Frid and Karlsen have used the strong trace result [38] with "non degenerate" flux to study the Initial Boundary Problem with zero-flux condition at the boundary.

5.1 The basic trace results

In this section we will rewrite our trace result in the kinetic form for the case of scalar conservation laws, and introduce Dubois and LeFloch's boundary condition (5.1) written in the kinetic form which plays an essential role in the proof of uniqueness.

$$[H(u) \cdot \hat{n}_s]^\tau - H(u_b) \cdot \hat{n} - \eta'(u_b)[(A(u) - A(u_b)) \cdot \hat{n}_s]^\tau \geq 0 \tag{5.1}$$

where B^τ means the trace of B on Γ.

We recall the kinetic formulation, Theorem 2.1.1, for the degenerate parabolic-hyperbolic equation. We notice that if we take $b(\xi) = c$ for all $\xi \in]-L, L[$ in the the equation (1.8), our kinetic formulation is the same as the result of Lions, Perthame and Tadmor in [21].

Theorem 5.1.1. *A function $u \in L^\infty(Q)$ with $|u| \leq L$ is a solution of (1.2) and (1.4) in Q if and only if there exists a nonnegative measure $m \in \mathcal{M}^+(Q\times]-L, L[)$ such that the related χ-function f defined by $f(t, x, \cdot) = \chi(u(t, x), \cdot)$ for almost every $(t, x, \xi) \in (Q\times]-L, L[)$ verifies:*

$$\partial_t f + \hat{a}(\xi) \cdot \nabla_x f = \partial_\xi m \tag{5.2}$$

in $Q\times]-L, L[$ with $\hat{a}(\xi) = A'(\xi)$.

From now on we will use the framework of the "regular deformable boundary" (see for instance Chen and Frid in [7]) given in section 1.4. Let us now give a simple version of Theorem 1.4.2 on a general domain Q given by Panov [26].

Theorem 5.1.2. *Let $\Omega \subset \mathbb{R}^d$ be a regular open set with C^2 boundary $\partial\Omega$ and the flux function A lie in $C^2(\mathbb{R})$. Consider any function $u \in L^\infty(Q)$ which verifies (1.2) and (1.4) in Q. For any function $\eta \in W^{1,1}(\mathbb{R})$ we consider $\overline{q}_\eta = (\eta, q)$ with the flux q verifying $q' = \eta' A'$. Then there exists $[\overline{q}_\eta(u)]^\tau \in [L^\infty(\partial Q)]^{d+1}$ such that, for every ∂Q regular deformation ψ and every compact* *set $K \subset\subset \partial Q$:*

$$\operatorname*{esslim}_{s\to 0} \int_K |\overline{q}_\eta(u(\psi(s,\hat{z}))) \cdot n_s(\hat{z}) - \overline{q}_\eta(u)^\tau(\hat{z}) \cdot n(\hat{z})| \, dV(\hat{z}) = 0,$$

where dV is the volume element of ∂Q. Moreover, there exists $a \cdot nf^\tau \in L^\infty(\partial Q \times]-L, L[)$ such that , for every ∂Q regular deformation ψ and every compact set $K \subset\subset \partial Q$:

$$\operatorname*{esslim}_{s\to 0} \int_K \int_{-L}^{L} |a(\xi) \cdot n_s(\hat{z}) f(\psi(s,\hat{z}),\xi) - a(\xi) \cdot n(\hat{z}) f^\tau(\hat{z},\xi)| \, d\xi dV(\hat{z}) = 0.$$

Notice that from Theorem 5.1.2, $[q(u) \cdot \hat{n}_s]^\tau$ and

$$F(u,u_b)^\tau := [(A(u) - A(u_b)) \cdot \hat{n}_s]^\tau \tag{5.3}$$

are well-defined in $L^\infty(\Gamma)$. Thus, we can rewrite (5.1) in the following kinetic form:

Lemma 5.1.1. *Let u be a solution of (1.2) and (1.4) and $u_b \in L^\infty(\Gamma)$. Then, the following are equivalent:*

- $[q(u) \cdot \hat{n}_s]^\tau - q(u_b) \cdot \hat{n} - \eta'(u_b) F(u,u_b)^\tau \geq 0$ *on Γ for every entropy flux $q' = \eta' A'$ with any convex η,*
- *There exists $\mu \in \mathcal{M}^+(\Gamma \times]-L, L[)$ such that $\hat{a}(\xi) \cdot \hat{n}(\hat{z})[f^\tau(\hat{z},\xi) - \chi(u_b(\hat{z}),\xi)] - \delta_{(\xi = u_b(\hat{z}))} F(u(\hat{z}), u_b(\hat{z}))^\tau = -\partial_\xi \mu(\hat{z},\xi)$ for every $(\hat{z},\xi) \in \Gamma \times]-L, L[$.*

Proof of Lemma 5.1.1: First the quantities $\hat{a} \cdot \hat{n} f^\tau$, $[q(u) \cdot \hat{n}_s]^\tau$, and $F(u,u_b)^\tau$ are well-defined thanks to Theorem 5.1.2. Then, for any convex function η, we have the following:

$$
\begin{aligned}
&[q(u)\cdot\hat{n}_s]^\tau - q(u_b)\cdot\hat{n} - \eta'(u_b)F(u,u_b)^\tau \\
&= \int_{-L}^{L} \eta'(\xi)\hat{a}(\xi)\cdot\hat{n}[f^\tau(\cdot,\xi)-\chi(u_b,\xi)] - \eta'(\xi)\delta_{(\xi=u_b)}F(u,u_b)^\tau d\xi
\end{aligned}
\tag{5.4}
$$

where $q' = \eta' A'$ and F is given in (5.3). Let us set:

$$
\begin{aligned}
\mu(\hat{z},\xi) = -\int_{-\infty}^{\xi} & \hat{a}(s)\cdot\hat{n}(\hat{z})[f^\tau(\hat{z},s)-\chi(u_b(\hat{z}),\xi)] \\
& -\eta'(s)\delta_{(s=u_b(\hat{z}))}F(u(\hat{z}),u_b(\hat{z}))^\tau ds
\end{aligned}
$$

for every $\hat{z}\in\Gamma$. It is obviously a measure. From (5.4), we have

$$
[q(u)\cdot\hat{n}_s]^\tau - q(u_b)\cdot\hat{n} - \eta'(u_b)F(u,u_b)^\tau \geq 0
$$

in Γ for every entropy flux q if and only if :

$$
-\int_{-L}^{L}\eta'(\xi)\partial_\xi\mu(\hat{z},d\xi) = \int_{-L}^{L}\eta''(\xi)\mu(\hat{z},d\xi)\geq 0,
$$

for any convex function η. This is equivalent to saying that μ is non-negative. The proof is complete. $\square$

The boundary condition (5.1) is well-defined in $L^\infty(\Gamma)$ and reached with L^1 convergence by Theorem 5.1.2.

5.2 The proof of uniqueness

Let us now introduce the well-posedness result of (1.2), (1.4), and (5.1) and we will give a simple proof.

Theorem 5.2.1. *Let $u_0(x)\in L^\infty(\Omega)$. Then, there exists a unique entropy solution $u\in L^\infty(]0,T[\times\Omega)$ verifying (1.2), (1.4), and (5.1). Moreover, if u*

and v are entropy solutions of (1.2), (1.4), and (5.1) with initial conditions $u(0,x) = u_0(x) \in L^\infty(\Omega)$ and $v(0,x) = v_0(x) \in L^\infty(\Omega)$, respectively, then, for a. e. $t \in]0,T[$,

$$\int_\Omega |u(t,x) - v(t,x)|dx \leq \int_\Omega |u_0(x) - v_0(x)|dx. \tag{5.5}$$

Notice that the existence for solution of (1.2), (1.4), and (5.1) is well-known by vanishing viscosity method. We next give the proof of Theorem 5.2.1. In fact, it is sufficient to show the uniqueness of Theorem 5.2.1. We first need to regularize the kinetic equation (2.6) with respect to the variables (t,x) by a convolution with mollified functions. This method was first initiated by Perthame [30] for the uniqueness proof of an initial value problem.

Step 1

Let u and v be solutions of (1.2) and (1.4). We set two χ functions f and g corresponding to solutions u and v, respectively by $f(t,x,\xi) = \chi(u(t,x),\xi)$ and $g(t,x,\xi) = \chi(v(t,x),\xi)$. We recall kinetic equations (2.6) for f and g respectively. From (2.6), there exist $m^1, m^2 \in \mathcal{M}^+(Q\times]-L,L[)$ such that

$$\partial_t f + \hat{a}(\xi)\cdot\nabla_x f = \partial_\xi m^1, \quad \textit{and} \quad \partial_t g + \hat{a}(\xi)\cdot\nabla_x g = \partial_\xi m^2. \tag{5.6}$$

for χ functions f, g respectively. In this step, we want to show the following inequality:

$$\begin{aligned}\int_\Omega \int_{-L}^{L} \partial_t |f(t,x,\xi) - g(t,x,\xi)|^2 d\xi dx \\ + \int_{\partial\Omega}\int_{-L}^{L} \hat{a}(\xi)\cdot\hat{n}(x)|f^\tau(t,x,\xi) - g^\tau(t,x,\xi)|^2 d\xi d\sigma \leq 0\end{aligned} \tag{5.7}$$

for $a.\ e.\ t \in]0,T[$ where $d\sigma$ is the volume element of $\partial\Omega$. We need first to regularize f and g with respect to the pair (t,x). We set $\epsilon = (\epsilon_1, \epsilon_2)$ and define ϕ_ϵ by

$$\phi_\epsilon(t,x) = \frac{1}{\epsilon_1}\phi_1(\frac{t}{\epsilon_1})\frac{1}{\epsilon_2^d}\phi_2(\frac{x}{\epsilon_2}),$$

where $\phi_1 \in \mathcal{C}_c^\infty(\mathbb{R})$ and $\phi_2 \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ verifying $\phi_j \geq 0$, $\int \phi_j = 1, j = 1,2, supp(\phi_1) \subset (0,1)$.

Let us present the following convenient notations:

- $f_\epsilon(t,x,\xi) = f(\cdot,\cdot,\xi) *_{(t,x)} \phi_\epsilon(t,x),\ g_\epsilon(t,x,\xi) = g(\cdot,\cdot,\xi) *_{(t,x)} \phi_\epsilon(t,x),$
- $m^1_\epsilon(t,x,\xi) = m^1(\cdot,\cdot,\xi) *_{(t,x)} \phi_\epsilon(t,x),\ m^2_\epsilon(t,x,\xi) = m^2(\cdot,\cdot,\xi) *_{(t,x)} \phi_\epsilon(t,x)$

where $*_{(t,x)}$ means the convolution in (t,x) and we extend f, g, m_1, m_2 to $\mathbb{R}^{d+1}$ by putting 0 on Q^c. We will next introduce a useful lemma which plays an important role in controling the part of entropy defect measures m^1, m^2 of u, v respectively and the proof is provided in Perthame [30].

Lemma 5.2.1. *Let m^1 and m^2 be nonnegative measures given in the Theorem 5.1.1. Then, the following holds.*

$$\lim_{\epsilon\to 0}\int_{-L}^{L} m^1_\epsilon(\cdot,\cdot,\xi)\delta_{(\xi=u)} * \phi_\epsilon + m^2_\epsilon(\cdot,\cdot,\xi)\delta_{(\xi=v)} * \phi_\epsilon d\xi = 0 \qquad (5.8)$$

in $\mathcal{D}'(Q)$.

Proof of (5.7): Consider a regular mollified function ϕ_ϵ as defined above. Let us denote a $\partial\Omega$ regular deformation by $\hat{\psi}$, and Ω_s denote the open subset of Ω, whose boundary is $\partial\Omega_s = \hat{\psi}(\{s\} \times \partial\Omega)$. Let us take the convolution of two kinetic equations (5.6). Then we subtract these two equations obtained above and multiply them by $f_\epsilon - g_\epsilon$, which yields:

$$\begin{aligned}\int_{\Omega_s}\int_{-L}^{L} &\partial_t|f_\epsilon(t,x,\xi) - g_\epsilon(t,x,\xi)|^2 \\ &+\hat{a}(\xi)\cdot\nabla_x|f_\epsilon(t,x,\xi) - g_\epsilon(t,x,\xi)|^2 d\xi d\sigma_s \\ = 2&\int_{\Omega_s}\int_{-L}^{L}\partial_\xi(m_\epsilon^1(t,x,\xi) - m_\epsilon^2(t,x,\xi))(f_\epsilon(t,x,\xi) - g_\epsilon(t,x,\xi))d\xi d\sigma_s\end{aligned} \tag{5.9}$$

for $a.\ e.\ s > 0$ where $d\sigma_s$ is the volume element of $\partial\Omega_s$. We now show that part of the defect measure is nonnegative. From the lemma 5.2.1, we obtain

$$\begin{aligned}&\lim_{\epsilon\to 0}\int_{\Omega_s}\int_{-L}^{L}\partial_\xi(m_\epsilon^1(\cdot,\cdot,\xi) - m_\epsilon^2(\cdot,\cdot,\xi))(f_\epsilon(\cdot,\cdot,\xi) - g_\epsilon(\cdot,\cdot,\xi))d\xi d\sigma_s \\ &= -\lim_{\epsilon\to 0}\int_{\Omega_s}\int_{-L}^{L}(m_\epsilon^1(\cdot,\cdot,\xi) - m_\epsilon^2(\cdot,\cdot,\xi))\partial_\xi(f_\epsilon(\cdot,\cdot,\xi) - g_\epsilon(\cdot,\cdot,\xi))d\xi d\sigma_s \\ &= -\lim_{\epsilon\to 0}\int_{\Omega_s}\int_{-L}^{L} m_\epsilon^1(\cdot,\cdot,\xi)\delta_{(\xi=v))} * \phi_\epsilon + m_\epsilon^2(\cdot,\cdot,\xi)\delta_{(\xi=u)} * \phi_\epsilon d\xi d\sigma_s \\ &\qquad \le 0.\end{aligned} \tag{5.10}$$

for $a.\ e.\ s > 0$. Let us apply the divergence theorem to (5.9) and take $\epsilon \to 0$, $s \to 0$. Then Theorem 5.1.2 gives the following inequality:

$$\begin{aligned}\int_{\Omega}\int_{-L}^{L} &\partial_t|f(t,x,\xi) - g(t,x,\xi)|^2 d\xi dx \\ &+\int_{\partial\Omega}\int_{-L}^{L}\hat{a}(\xi)\cdot\hat{n}(x)|f^\tau(t,x,\xi) - g^\tau(t,x,\xi)|^2 d\xi d\sigma(x) \le 0\end{aligned} \tag{5.11}$$

for $a.\ e.\ t \in]0,T[$. □

Step 2

Next we need to show that the second part of (5.11) is nonnegative. We set the measures given in Lemma 5.1.1, corresponding to f, g, by μ_1, μ_2

respectively such that there exists $\mu_1, \mu_2 \in \mathcal{M}^+(\Gamma\times]-L, L[)$ verifying:

$$\begin{array}{l}\hat{a}(\xi)\cdot\hat{n}(\hat{z})[f^\tau(\hat{z},\xi)-\chi(u_b(\hat{z}),\xi)]-\delta_{(\xi=u_b(\hat{z}))}F(u(\hat{z}),u_b(\hat{z}))^\tau=-\partial_\xi\mu_1(\hat{z},\xi)\\ \hat{a}(\xi)\cdot\hat{n}(\hat{z})[g^\tau(\hat{z},\xi)-\chi(u_b(\hat{z}),\xi)]-\delta_{(\xi=u_b(\hat{z}))}F(v(\hat{z}),u_b(\hat{z}))^\tau=-\partial_\xi\mu_2(\hat{z},\xi)\end{array} \tag{5.12}$$

for $(\hat{z},\xi)\in\Gamma\times]-L,L[$. We introduce the following fact which will be used later:

$$\begin{aligned}&\hat{a}\cdot\hat{n}|f^\tau-g^\tau|^2\\ &=\hat{a}\cdot\hat{n}(f^\tau-\chi(u_b,\xi))sgn(\xi-u_b)-2\hat{a}\cdot\hat{n}(f^\tau-\chi(u_b,\xi))(g^\tau-\chi(u_b,\xi))\\ &\qquad\qquad+\hat{a}\cdot\hat{n}(g^\tau-\chi(u_b,\xi))sgn(\xi-u_b)\\ &=\hat{a}\cdot\hat{n}(f^\tau-\chi(u_b,\xi))[sgn(\xi-u_b)-g^\tau+\chi(u_b,\xi)]\\ &\qquad\qquad+\hat{a}\cdot\hat{n}(g^\tau-\chi(u_b,\xi))[sgn(\xi-u_b)-f^\tau+\chi(u_b,\xi)]\end{aligned} \tag{5.13}$$

where

$$sgn(\xi-u_b)=\begin{cases}\mathbf{1} & \text{if} \quad u_b\le\xi<L\\ -\mathbf{1} & \text{if} -L<\xi<u_b\\ 0 & \text{otherwise}\end{cases}$$

Let us denote $\alpha(\xi)=sgn(\xi-u_b)-g^\tau+\chi(u_b,\xi)$ and $\beta(\xi)=sgn(\xi-u_b)-f^\tau+\chi(u_b,\xi)$. Combining (5.12), (5.13) and using integration by part give the following equality:

$$\begin{aligned}
&\int_{\partial\Omega}\int_{-L}^{L} \hat{a}(\xi)\cdot\hat{n}(\hat{z})|f^\tau(\hat{z},\xi)-g^\tau(\hat{z},\xi)|^2 d\xi d\sigma \\
&=\int_{\partial\Omega}\int_{-L}^{L} \hat{a}(\xi)\cdot\hat{n}(\hat{z})[f^\tau(\hat{z},\xi)-\chi(u_b(\hat{z}),\xi)]\alpha(\xi) d\xi d\sigma \\
&\quad+\int_{\partial\Omega}\int_{-L}^{L} \hat{a}(\xi)\cdot\hat{n}(\hat{z})[g^\tau(\hat{z},\xi)-\chi(u_b(\hat{z}),\xi)]\beta(\xi) d\xi d\sigma \\
&=\int_{\partial\Omega}(\int_{-L}^{u_b}+\int_{u_b}^{L})(-\partial_\xi\mu_1(\hat{z},\xi)\alpha(\xi)-\partial_\xi\mu_2(\hat{z},\xi)\beta(\xi))d\xi d\sigma \\
&=\int_{\partial\Omega}\int_{-L}^{u_b}\mu_1(\hat{z},\xi)\nu_2 d\xi d\sigma-\mu_1(u_b^-)\alpha(u_b^-) \\
&\quad+\int_{\partial\Omega}\int_{u_b}^{L}\mu_1(\hat{z},\xi)\nu_2 d\xi d\sigma+\mu_1(u_b^+)\alpha(u_b^+) \\
&\qquad+\int_{\partial\Omega}\int_{-L}^{u_b}\mu_2(\hat{z},\xi)\nu_1 d\xi d\sigma-\mu_2(u_b^-)\beta(u_b^-) \\
&\qquad\quad+\int_{\partial\Omega}\int_{u_b}^{L}\mu_2(\hat{z},\xi)\nu_1 d\xi d\sigma+\mu_2(u_b^+)\beta(u_b^+),
\end{aligned} \tag{5.14}$$

where ν_1, ν_2 are Young measures defined by $\partial_\xi f^\tau = \delta(\xi)-\nu_1$ and $\partial_\xi g^\tau = \delta(\xi)-\nu_2$. Notice that $\alpha(u_b^+)\geq 0$, $\beta(u_b^+)\geq 0$ and $\alpha(u_b^-)\leq 0$, $\beta(u_b^-)\leq 0$. Thus, (5.14) is nonnegative.

Step 3

We will finish the proof of Theorem 1.4.2 by combining step 1 and step 2. Since the second part of (5.11) is nonnegative, integrating the first part of (5.11) implies that: for fix $0 < s < t$,

$$\int_\Omega\int_{-L}^{L}|f(t,x,\xi)-g(t,x,\xi)|^2 d\xi dx \leq \int_\Omega\int_{-L}^{L}|f(s,x,\xi)-g(s,x,\xi)|^2 d\xi dx$$

Therefore, Theorem 5.1.2 implies that, as s goes to 0,

$$\int_\Omega |u(t,x)-v(t,x)|dx \leq \int_\Omega |u_0(x)-v_0(x)|dx$$

for $a.\ e.\ t \in]0, T[$. Indeed, $|f(t,x,\xi) - g(t,x,\xi)|^2 = |f(t,x,\xi) - g(t,x,\xi)|$. The proof is complete. $\square$

Appendices

Proof of Lemma 3.2.1: Since $p_0 \in E$, there exists $V_r = \{p \in P||p - p_0| < r\} \subset E$. Let us denote $L_r = \bigcap_{p \in V_r} L(p)$. Then, by the definition of L and $L(p)$, $L_r \subset L$ and

$$r \longmapsto V_r$$

is decreasing in the subset sense. Obviously we can take $r_0 > 0$ such that L_r have the same dimension for all $r \in (0, r_0]$. Let us denote such a linear subspace by W. Then, $\dim W \leq l$.

Next we need to claim $\dim W \geq l$. Since $W = L_{r_0} \subset L(p) \subset L$ for all $p \in V_{r_0}$, we show $L(p) = L$ for all $p \in V_{r_0}$ thanks to the claim.

Let us now prove the claim. Suppose that $s = \dim W < l$. From Theorem 2.2.1, we know that $(p, q) \longmapsto \mu^{pq}$ is continuous which means that: for any $\epsilon > 0$, there exists $0 < \delta < r_0$ such that $p \in V_\delta$ implies

$$var(\mu^{pq} - \mu^{p_0 q})(K \times S) < \epsilon \tag{15}$$

for a compact subset $K \subset \Omega$. By the assumption, we know $l - s > 0$. So, we can take $\{p_i\}_{i=1}^m$ such that

$$W = L(p_1) \subset L(p_2) \subset ... \subset L \tag{16}$$

From the definition of $L(P_i)$ and (15), we obtain that *supp* $\mu^{p_i q} \subset \Omega \times L(p_i)$ which yields

$$var(\mu^{p_0 q}(K \times (L(p_i))^c) < \epsilon \tag{17}$$

(16) and $W^c \subset L^c \bigcup_{i=1}^{s} (L(p_i))^c$, which gives,

$$var(\mu^{p_0 q})(K \times W^c) \leq var(\mu^{p_0 q})(K \times L^c) + \sum_{i=1}^{s} var(\mu^{p_0 q})(K \times (L(p_i))^c) \leq m\epsilon$$

It follows that $var(\mu^{p_0 q})(K \times W^c) = 0$ and *supp* $\mu^{p_0 q} \subset \Omega \times W$. Thus, the definition of L deduces the contradiction. Therefore $s = m$ and $W = L$. The proof is complete. $\square$

Proof of Lemma 3.4.3: Assume that $\beta(s)g(s)$ converges strongly to $\beta_0 g^\tau$ in $L^1_{\text{loc}}(]-r_\alpha, r_\alpha[^d \times J)$ when $s \to 0$. Then, there exists a sequence s_n converging to 0 such that $\beta(s_n, \hat{w}, \xi) g(s_n, \hat{w}, \xi)$ converges to $\beta_0(\hat{w}, \xi) g^\tau(\hat{w}, \xi)$ for almost every $(\hat{w}, \xi) \in]-r_\alpha, r_\alpha[^d \times J$. Let us fix such a point $(\hat{w}, \xi)$. If $\beta_0(\hat{w}, \xi)$ is different from 0 then, since β is continuous, $g(s_n, \hat{w}, \xi)$ converges to $g^\tau(\hat{w}, \xi)$. But g is a χ-function, so for every n, $\text{sign}(\xi) g(s_n, \hat{w}, \xi)$ is equal to 1 or 0. so its limit is 1 or 0. This shows that for almost every $(\hat{w}, \xi) \in]-r_\alpha, r_\alpha[^d \times J$ we have $\beta_0(\hat{w}, \xi) g^\tau(\hat{w}, \xi)$ is equal to $\text{sign}(\xi)\beta_0(\hat{w}, \xi)$ or 0.

Conversely, assume that for almost every $(\hat{w}, \xi)$ in $]-r_\alpha, r_\alpha[^d \times J$, we have $\beta_0(\hat{w}, \xi) g^\tau(\hat{w}, \xi)$ equal to $\text{sign}(\xi)\beta_0(\hat{w}, \xi)$ or 0. Since g is a χ-function, we have:

$$\int_J \int_{]-r_\alpha,r_\alpha[^d} |\beta(s,\hat{w},\xi)g(s,\hat{w},\xi)|^2 \, d\hat{w}\, d\xi$$
$$= \int_J \int_{]-r_\alpha,r_\alpha[^d} |\beta(s,\hat{w},\xi)|^2 \mathrm{sign}(\xi) g(s,\hat{w},\xi)\, d\hat{w}\, d\xi.$$

But β is continuous, so for every $(\hat{w},\xi)$ in $]-r_\alpha,r_\alpha[^d \times J$, $\beta(s,\hat{w},\xi)\mathrm{sign}(\xi)$ converges to $\beta_0(\hat{w},\xi)\mathrm{sign}(\xi)$. Since $\beta(s)g(s)$ converges weakly to $\beta_0 g^\tau$ we have that:

$$\lim_{s\to 0} \int_J \int_{]-r_\alpha,r_\alpha[^d} |\beta(s,\hat{w},\xi)|^2 \mathrm{sign}(\xi) g(s,\hat{w},\xi)\, d\hat{w}\, d\xi$$
$$= \int_J \int_{]-r_\alpha,r_\alpha[^d} |\beta_0(\hat{w},\xi)|^2 \mathrm{sign}(\xi) g^\tau(\hat{w},\xi)\, d\hat{y}\, d\xi.$$

From the hypothesis:

$$\beta_0^2 \mathrm{sign}(\xi) g^\tau = \beta_0^2 |g^\tau|^2.$$

Indeed, the equality is trivial at the points where $\beta_0(\hat{w},\xi) = 0$. And if $\beta_0(\hat{w},\xi) \neq 0$, then the hypothesis gives that $g^\tau(\hat{w},\xi)$is equal to $\mathrm{sign}(\xi)$ or 0 at this point. The equality is verified for both cases. Altogether, this shows that:

$$\lim_{s\to 0} \int_J \int_{]-r_\alpha,r_\alpha[^d} |\beta(s,\hat{w},\xi)g(s,\hat{w},\xi)|^2 \, d\hat{w}\, d\xi = \int_J \int_{]-r_\alpha,r_\alpha[^d} |\beta_0(\hat{w},\xi)g^\tau(\hat{w},\xi)|^2 \, d\hat{w}\, d\xi.$$

Hence $\beta(s)g(s)$ converges weakly in L^2 to $\beta_0 g^\tau$ and $\|\beta(s)g(s)\|_{L^2}$ converges to $\|\beta_0 g^\tau\|_{L^2}$. Hence the convergence holds strongly. $\square$

Proof of Lemma 3.3.1: For every integer N, we denote

$$M_\delta^N(\hat{w}) = \frac{1}{\delta^d} \hat{m}\left(]0, N\delta[\times (\hat{w} +]-N\delta, N\delta[^d) \times I\right).$$

Since M_δ^N is nonnegative, the L^1 norm of M_δ^N is :

$$\begin{aligned}
&\int_{]-r_\alpha,r_\alpha[^d} M_\delta^N(\hat{w})\,d\hat{w}\\
&= \int_{]-r_\alpha,r_\alpha[^d} \frac{1}{\delta^d}\int_0^{N\delta}\int_{]-N\delta,N\delta[^d}\int_I \hat{m}(s,\hat{w}+\hat{z},\xi)\,d\xi\,d\hat{z}\,ds\,d\hat{w}\\
&\le \frac{1}{\delta^d}\int_{]-N\delta,N\delta[^d}\int_0^{N\delta}\int_I\int_{]-r_\alpha-N\delta,r_\alpha+N\delta[^d} \hat{m}(s,\hat{w},\xi)\,d\hat{w}\,d\xi\,ds\,d\hat{z}.
\end{aligned}$$

We denote abusively $\hat{m}(ds,d\hat{z},d\xi)=\hat{m}(s,\hat{z},\xi)\,ds\,d\hat{z}\,d\xi$ in this computation as if it was a function. This calculation is still correct since we just use the Fubini Theorem and a linear change of variable which are valid for measures. The last inequality can be written as:

$$\begin{aligned}
&\int_{]-r_\alpha,r_\alpha[^d} M_\delta^N(\hat{w})\,d\hat{w}\\
&\le \frac{1}{\delta^d}\int_{]-N\delta,N\delta[^d} \hat{m}(]0,N\delta[\times]-r_\alpha-N\delta,r_\alpha+N\delta[^d\times I)\,d\hat{z}\\
&\le N\hat{m}(]0,N\delta[\times]-r_\alpha-N\delta,r_\alpha+N\delta[^d\times I).
\end{aligned}$$

By monotone convergence, since $\bigcap_{\delta>0}]0,N\delta[=\emptyset$, this converges to 0 when δ converges to 0. Finally the L^1 norm of M_δ^N converges to 0 so there exists a subsequence δ_n and a set $\mathcal{E}_N\subset]-r_\alpha,r_\alpha[^d$ with $\mathcal{L}(]-r_\alpha,r_\alpha[^d\backslash\mathcal{E}_N)=0$ such that for every $\hat{w}\in\mathcal{E}_N$ $M_{\delta_n}^N(\hat{w})$ converges to 0 when δ_n goes to 0. By diagonal extraction, we can choose δ_n such that for every integer N and every $\hat{w}\in\mathcal{E}_N$, $M_{\delta_n}^N(\hat{w})$ converges to 0. This sequence δ_n with subset $\mathcal{E}=\cap_N\mathcal{E}_N$ verifies the required condition. $\square$

Proof of Lemma 3.3.2: For every integer N we denote:

$$F^N_{\delta_n}(\hat{w}) = \int_I \int_{]-N,N[^d} |g^\tau(\hat{w},\xi) - g^\tau(\hat{w} + \delta_n \underline{\hat{w}}, \xi)| \, d\underline{\hat{w}} \, d\xi.$$

Since $g^\tau \in L^\infty(]-r_\alpha, r_\alpha[^d \times I)$, the L^1 norm of this function goes to zero as n tends to ∞ so there exists a subsequence still denoted δ_n and a subset $\mathcal{E}'_N \subset \mathcal{E}$ with $\mathcal{L}(]-r_\alpha, r_\alpha[^d \backslash \mathcal{E}'_N) = 0$ such that for every $\hat{w} \in \mathcal{E}'_N$, $F^N_{\delta_n}(\hat{w})$ converges to 0 when n tends to infinity. By diagonal extraction we can find a subsequence such that this holds true for every N. Then this subsequence and $\mathcal{E}' = \cap_N \mathcal{E}'_N$ fulfill the required condition for the first limit. $\square$

Proof of Lemma 4.3.1: Since f_n is a χ-function we have

$$\partial_\xi f_n = \delta(\xi) - \delta(\xi - u_n) \qquad \text{in } \ I. \tag{18}$$

So at the limit $\partial_\xi f \in L^\infty(\mathcal{M}(I))$, which means that for almost every $\hat{w} \in \mathcal{O}$, $f(\hat{w}, \cdot)$ lies in $BV(I)$.

Now, if f_n converges strongly, the same holds for u_n. If u_n converges strongly, then its young measure $\delta(\xi - u_n)$ converges to $\delta(\xi - u)$. Hence:

$$\partial_\xi f = \delta(\xi) - \delta(\xi - u) \qquad \text{in } \ I.$$

This ensures that f is a χ-function. Finally, if f is a χ-function, in particular, $\text{sgn}(\xi) f = f^2$ so $\|f_n\|_{L^2_{\text{loc}}(\mathcal{O}\times I)}$ converges to $\|f\|_{L^2_{\text{loc}}(\mathcal{O}\times I)}$. This provides the strong convergence of f_n in $L^2_{\text{loc}}(\mathcal{O} \times I)$ and then in $L^1_{\text{loc}}(\mathcal{O} \times I)$. $\square$

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